

One Procedure to Find the Inverse of a Matrix

Only square matrices with non-zero determinants have inverses. Consequently any matrix A considered in this paper is assumed to be square with $\det(A) \neq 0$.

Two matrices A and B are inverses of each other if $AB = I$ and $BA = I$ where I is the identity matrix of the same order as A and B.

If A and I both have order $n \times n$, to adjoin the identity matrix I to the square matrix A means to form a new matrix of order $n \times 2n$ by writing the columns of I to the right of the columns of A. A slight abuse of notation permits us to symbolize this new matrix as $[A:I]$. Because the vertical dotted line is difficult to represent on computers, it is frequently omitted.

The Elementary Row Operations are:

1. Interchange two rows
2. Multiply a row by a non-zero constant and replace that row with the product.
3. Add a multiple of a row to another row and replace one but not both of the rows with that sum.

A) General Overview of the Procedure

Steps to find the inverse of a matrix A

1. Adjoin the identity matrix to A to obtain the $n \times 2n$ matrix $[A:I]$
2. Use elementary row operations on $[A:I]$ to convert A to the identity I.
3. The identity matrix on the right of $[A:I]$ will have been converted to the inverse of A.

To summarize:

$$[A:I] \xrightarrow{\text{a series of elementary row operations}} [I:A^{-1}]$$

B) More Detail of the Procedure

Steps to find the inverse of a matrix A

1. Adjoin the identity matrix to A to obtain the $n \times 2n$ matrix $[A:I]$
2. Use elementary row operations on $[A:I]$ to convert A to the identity I.
 - a. Get a 1 in the 1,1 position.
 - b. Use the 1 in the 1,1 position to get zeros everywhere else in the column.
 - c. Get a 1 in the 2,2 position.
 - d. Use the 1 in the 2,2 position to get zeros everywhere else in the columncontinue this process until you reach the end of the diagonal (the n,n position)
Get a 1 in the n,n position.
Use the 1 in the n,n position to get zeros everywhere in the column.
3. The identity matrix on the right of $[A:I]$ will have been converted to the inverse of A.

To summarize:

$$[A:I] \xrightarrow{\text{a series of elementary row operations}} [I:A^{-1}]$$

C) Computational Details of the Procedure

Steps to find the inverse of a matrix A

1. Adjoin the identity matrix to A to obtain the $n \times 2n$ matrix $[A:I]$

2. Use elementary row operations on $[A:I]$ to convert A to the identity I.

a. Get a 1 in the 1,1 position.

Do this by multiplying Row 1 by the multiplicative inverse (reciprocal) of the 1,1 entry

$$\text{Symbolically: } \frac{1}{a_{11}} R_1 \longrightarrow R_1$$

b. Use the 1 in the 1,1 position to get zeros everywhere else in the column.

i) To get a zero in the 2,1 position: Add a multiple of Row 1 to Row 2 and replace Row 2 with that sum. The multiplier to use is the additive inverse of the 2,1 entry. This does not change Row 1.

$$\text{Symbolically: } -a_{21} R_1 + R_2 \longrightarrow R_2$$

ii) To get a zero in the 3,1 position: Add a multiple of Row 1 to Row 3 and replace Row 3 with that sum. The multiplier to use is the additive inverse of the 3,1 entry. This does not change Row 1.

$$\text{Symbolically: } -a_{31} R_1 + R_3 \longrightarrow R_3$$

Continue this process until there is a 0 everywhere in the column except the 1,1 entry.

c. Get a 1 in the 2,2 position.

Do this by multiplying Row 2 by the multiplicative inverse (reciprocal) of the 2,2 entry

$$\text{Symbolically: } \frac{1}{a_{22}} R_2 \longrightarrow R_2$$

d. Use the 1 in the 2,2 position to get zeros everywhere else in the column

i) To get a zero in the 1,2 position: Add a multiple of Row 2 to Row 1 and replace Row 1 with that sum. The multiplier to use is the additive inverse of the 1,2 entry. This does not change Row 2.

$$\text{Symbolically: } -a_{12} R_2 + R_1 \longrightarrow R_1$$

ii) To get a zero in the 3,2 position: Add a multiple of Row 2 to Row 3 and replace Row 3 with that sum. The multiplier to use is the additive inverse of the 3,2 entry. This does not change Row 2.

$$\text{Symbolically: } -a_{32} R_2 + R_3 \longrightarrow R_3$$

Continue this process until there is a 0 everywhere in the column except the 2,2 entry.

continue this process until you reach the end of the diagonal (the n,n position)

Get a 1 in the n,n position.

Do this by multiplying Row n by the multiplicative inverse (reciprocal) of the n,n entry

$$\text{Symbolically: } \frac{1}{a_{nn}} R_n \longrightarrow R_n$$

Use the 1 in the n,n position to get zeros everywhere in the column.

i) To get a zero in the 1,n position: Add a multiple of Row n to Row 1 and replace Row 1 with that sum. The multiplier to use is the additive inverse of the 1,n entry. This does not change Row n.

$$\text{Symbolically: } -a_{1n} R_n + R_1 \longrightarrow R_1$$

ii) To get a zero in the 2,n position: Add a multiple of Row n to Row 2 and replace Row 2 with that sum. The multiplier to use is the additive inverse of the 2,n entry. This does not change Row n.

$$\text{Symbolically: } -a_{2n} R_n + R_2 \longrightarrow R_2$$

Continue this process until there is a 0 everywhere in the column except the n,n entry.

3. The identity matrix on the right of $[A:I]$ will have been converted to the inverse of A.

To summarize:

$$[A:I] \xrightarrow{\text{a series of elementary row operations}} [I:A^{-1}]$$

D) Examples of the Procedure

Problem:

Find the inverse of $\begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{1}{2}R_1 \rightarrow R_1} \begin{bmatrix} 1 & 2 & \frac{1}{2} & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix} \xrightarrow{-4R_1 + R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & \frac{1}{2} & 0 \\ 0 & -3 & -2 & 1 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{3}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \xrightarrow{-2R_2 + R_1 \rightarrow R_1} \begin{bmatrix} 1 & 0 & -\frac{5}{6} & \frac{2}{3} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\text{Therefore } A^{-1} = \begin{bmatrix} -\frac{5}{6} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \text{ which may also be written as } \frac{1}{3} \begin{bmatrix} -5 & 2 \\ 2 & -1 \end{bmatrix}$$

which may also be written as $\left(-\frac{1}{6}\right) \begin{bmatrix} 5 & -4 \\ -4 & 2 \end{bmatrix}$

This last expression matches the formula for a 2 X 2 matrix as presented on Page 555.

Problem:

Find the inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

Solution:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 3 & 7 & 9 & 0 & 0 & 0 \\ -1 & -4 & -7 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\begin{matrix} -3R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \end{matrix}]{} \begin{bmatrix} 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & -2 & -5 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\begin{matrix} -2R_2 + R_1 \\ 2R_2 + R_3 \end{matrix}]{} \begin{bmatrix} 1 & 0 & -4 & 7 & -2 & 0 \\ 0 & 1 & 3 & -3 & 1 & 0 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix} \xrightarrow[\begin{matrix} 4R_3 + R_1 \rightarrow R_1 \\ -3R_3 + R_2 \rightarrow R_2 \end{matrix}]{} \begin{bmatrix} 1 & 0 & 0 & -13 & 6 & 4 \\ 0 & 1 & 0 & 12 & -5 & -3 \\ 0 & 0 & 1 & -5 & 2 & 1 \end{bmatrix}$$

The inverse of $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$ is $\begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$

Check it by computing both $\begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix} \begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix}$ and $\begin{bmatrix} -13 & 6 & 4 \\ 12 & -5 & -3 \\ -5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 3 & 7 & 9 \\ -1 & -4 & -7 \end{bmatrix}$

Both products will be the 3 X 3 identity matrix.