

Polynomial Function – Analysis and Graphing

Problem: Analyze and Sketch the Graph of the function f whose rule is

$$f(x) = 4x^4 + 16x^3 + 25x^2 + 21x + 9$$

Analysis:

The function f is a polynomial function. Therefore its graph will be a smooth continuous curve with no gaps or sharp corners.

The degree of f is four. Therefore its graph tries to cross the x -axis four times and tries to exhibit three turning points (humps).

The degree of f is even. Therefore its graph need not have an x -intercept.

Because f is a polynomial function the leading term will dominate when x is far from the origin. Therefore

$$\text{as } x \rightarrow \infty, f(x) \rightarrow \infty \text{ and as } x \rightarrow -\infty, f(x) \rightarrow \infty$$

We now have enough information to expect the graph of f to look like some variation of the graph in Fig. 1. However, we must recognize that the number of x -intercepts and the number of turning points might be very different than shown in Fig. 1.

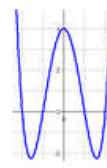


Fig. 1

To determine the x -intercepts of the graph of f we must find the zeros of f .

To find the zeros of f (as with all functions) we must solve the equation resulting from $f(x) = 0$.

In this case we must solve the equation $0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9$

We cannot solve this equation directly, so we turn to the rational zeros theorem to find the possible rational zeros of the function f .

If $\frac{p}{q}$ is a rational zero of the function f , then p is a divisor of the constant term 9 and q is a divisor of the

leading coefficient 4. So $p \in N = \{\pm 1, \pm 3, \pm 9\}$ and $q \in D = \{\pm 1, \pm 2, \pm 4\}$ and then $\frac{p}{q}$ must be in the

set of all fractions which may be constructed by selecting numerators from N and selecting denominators

from D . Therefore $\frac{p}{q} \in \left\{ \pm 1, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{9}{4} \right\} = K$. Thus if the function f has a

rational zero (a rational x -intercept) then it must be one of the above 18 rational numbers in the set labeled K .

To avoid testing each of these, it is wise at this point to use a graphing utility such as Omnigraph or a TI 83 to produce a graph of f as shown in Fig. 2. The graph in Fig. 2 makes it clear that the only reasonable options for rational zeros must be in the interval $(-2, -1)$. The only element of the

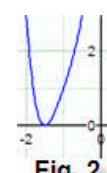


Fig. 2

set K in this interval is the number $-\frac{3}{2}$. We must now test $-\frac{3}{2}$ to see if it is a zero of the

function f . That is we must determine if $f\left(-\frac{3}{2}\right) = 0$. We can avoid substituting $-\frac{3}{2}$ into the rule for f

by recalling that the following statements are equivalent.

- 1) $-\frac{3}{2}$ is a zero of f
- 2) $x - \left(-\frac{3}{2}\right) = x + \frac{3}{2}$ is a factor of $4x^4 + 16x^3 + 25x^2 + 21x + 9$

But even the prospect of performing this division with a fraction in the divisor is not appealing. However, if we note that

$x + \frac{3}{2}$ is a factor of $4x^4 + 16x^3 + 25x^2 + 21x + 9$ if and only if

$2x + 3$ is a factor of $4x^4 + 16x^3 + 25x^2 + 21x + 9$

So we perform the later division as shown in Fig. 3 to obtain a quotient of $2x^3 + 5x^2 + 5x + 3$ and a remainder of 0. This shows that

$$0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9 = (2x + 3)(2x^3 + 5x^2 + 5x + 3)$$

And now The Zero Factor Property implies

$$2x + 3 = 0 \text{ OR } 2x^3 + 5x^2 + 5x + 3 = 0$$

$$x = -\frac{3}{2} \text{ OR } 2x^3 + 5x^2 + 5x + 3 = 0$$

Thus $-\frac{3}{2}$ is a zero of f . To determine the other zeros of f we must solve the equation $2x^3 + 5x^2 + 5x + 3 = 0$.

Note that is the same as finding the zeros of the function g whose rule is

$$g(x) = 2x^3 + 5x^2 + 5x + 3.$$

The graph of g is shown in red superimposed over the graph of f in Fig. 4.

It appears from Fig. 4 that $-\frac{3}{2}$ is a zero of g . So it appears that

$2x + 3$ is a factor of $2x^3 + 5x^2 + 5x + 3$.

This fact is verified in the division shown in Fig.5.

This division yields the following

$$0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9 = (2x + 3)(2x^3 + 5x^2 + 5x + 3) = (2x + 3)^2(x^2 + x + 1)$$

And again we invoke The Zero Factor Property to conclude that

$$(2x + 3)^2 = 0 \text{ OR } x^2 + x + 1 = 0$$

$$x = -\frac{3}{2} \text{ OR } x^2 + x + 1 = 0$$

So $-\frac{3}{2}$ is a zero with multiplicity 2.

The other zeros are found with the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

We have now determined that the four zeros of the original function f are:

$$-\frac{3}{2} \text{ with multiplicity 2, and the two complex conjugates } \frac{-1 + \sqrt{3}i}{2}, \text{ and } \frac{-1 - \sqrt{3}i}{2}$$

We may now conclude that the graph of f will intersect the x -axis at $-\frac{3}{2}$ but will not cross the

axis. It has no other x -intercepts. This permits us to label the one x -intercept and together with the information in Fig. 1 we may produce the graph shown in Fig. 6.

$$\begin{array}{r} 2x^3 + 5x^2 + 5x + 3 \\ 2x + 3 \overline{) 4x^4 + 16x^3 + 25x^2 + 21x + 9} \\ \underline{4x^4 + 6x^3} \\ 10x^3 + 25x^2 + 21x + 9 \\ \underline{10x^3 + 15x^2} \\ 10x^2 + 21x + 9 \\ \underline{10x^2 + 15x} \\ 6x + 9 \\ \underline{6x + 9} \\ 0 \end{array}$$

Fig. 3

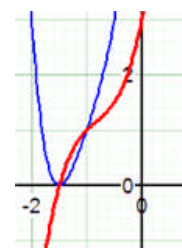


Fig. 4

$$\begin{array}{r} x^2 + x + 1 \\ 2x + 3 \overline{) 2x^3 + 5x^2 + 5x + 3} \\ \underline{2x^3 + 3x^2} \\ 2x^2 + 5x + 3 \\ \underline{2x^2 + 3x} \\ 2x + 3 \\ \underline{2x + 3} \\ 0 \end{array}$$

Fig. 6

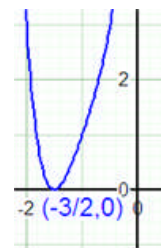


Fig. 6