Polynomial Function – Analysis and Graphing

Problem: Analyze and Sketch the Graph of the function f whose rule is

$$f(x) = 4x^4 + 16x^3 + 25x^2 + 21x + 9$$

Analysis:

The function f is a polynomial function. Therefore its graph will be a smooth continuous curve with no gaps or sharp corners.

The degree of f is four. Therefore its graph tries to cross the x-axis four times and tries to exhibit three turning points (humps).

The degree of f is even. Therefore its graph need not have an x-intercept.

Because f is a polynomial function the leading term will dominate when x is far from the origin. Therefore

$$\text{as } x \to \infty,\ f(x) \to \infty \ \text{ and as } x \to -\infty,\ f(x) \to \infty$$

We now have enough information to expect the graph of f to look like some variation of the graph in Fig. 1. However, we must recognize that the number of x-intercepts and the number of turning points might be very different than shown in Fig. 1.

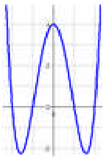

Fig. 1

To determine the x-intercepts of the graph of f we must find the zeros of f.

To find the zeros of f (as with all functions) we must solve the equation resulting from f(x) = 0.

In this case we must solve the equation $0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9$

We cannot solve this equation directly, so we turn to the rational zeros theorem to find the possible rational zeros of the function f.

If $\frac{p}{q}$ is a rational zero of the function f, then p is a divisor of the constant term 9 and q is a divisor of the leading coefficient 4. So $p \in N = \{\pm 1, \pm 3, \pm 9\}$ and $q \in D = \{\pm 1, \pm 2, \pm 4\}$ and then $\frac{p}{q}$ must be in the set of all fractions which may be constructed by selecting numerators from N and selecting denominators from D. Therefore $\frac{p}{q} \in \left\{\pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}, \pm\frac{1}{4}, \pm\frac{1}{4}, \pm\frac{1}{4}\right\} = K$. Thus if the function f has a rational zero (a rational x-intercept) then it must be one of the above 18 rational numbers in the set labeled K.

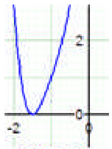

Fig. 2

To avoid testing each of these, it is wise at this point to use a graphing utility such as Omnigraph or a TI 83 to produce a graph of f as shown in Fig. 2. The graph in Fig. 2 makes it clear that the only reasonable options for rational zeros must be in the interval (-2, -1). The only element of the set K in this interval is the number $-\frac{3}{2}$. We must now test $-\frac{3}{2}$ to see if it is a zero of the function f. That is we must determine if $f\left(-\frac{3}{2}\right) = 0$. We can avoid substituting $-\frac{3}{2}$ into the rule for f by recalling that the following statements are equivalent.

1) $-\frac{3}{2}$ is a zero of f

2) $x - \left(-\frac{3}{2}\right) = x + \frac{3}{2}$ **is a factor of** $4x^4 + 16x^3 + 25x^2 + 21x + 9$

But even the prospect of performing this division with a fraction in the divisor is not appealing. However, if we note that

$x+\frac{3}{2}$ **is a factor of** $4x^4+16x^3+25x^2+21x+9$ if and only if

$2x+3$ **is a factor of** $4x^4+16x^3+25x^2+21x+9$

So we perform the later division as shown in Fig. 3 to obtain a quotient of $2x^3+5x^2+5x+3$ and a remainder of 0. This shows that

$$0=4x^4+16x^3+25x^2+21x+9=(2x+3)(2x^3+5x^2+5x+3)$$

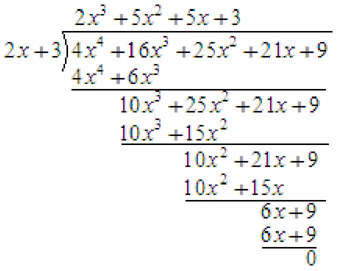

Fig. 3

And now The Zero Factor Property implies

$$2x+3=0 \text{ OR } 2x^3+5x^2+5x+3=0$$

$$x=-\frac{3}{2} \text{ OR } 2x^3+5x^2+5x+3=0$$

Thus $-\frac{3}{2}$ is a zero of f. To determine the other zeros of f we must solve the equation $2x^3+5x^2+5x+3=0$. Note that is the same as finding the zeros of the function g whose rule is $g(x)=2x^3+5x^2+5x+3$.

The graph of g is shown in red superimposed over the graph of f in Fig. 4.

It appears from Fig. 4 that $-\frac{3}{2}$ is a zero of g. So it appears that

$2x+3$ **is a factor of** $2x^3+5x^2+5x+3$.

Fig. 4

This fact is verified in the division shown in Fig.5.

This division yields the following

$$0=4x^4+16x^3+25x^2+21x+9=(2x+3)(2x^3+5x^2+5x+3)=(2x+3)^2(x^2+x+1)$$

And again we invoke The Zero Factor Property to conclude that

$$(2x+3)^2=0 \text{ OR } x^2+x+1=0$$

$$x=-\frac{3}{2} \text{ OR } x^2+x+1=0$$

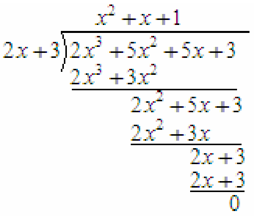

Fig. 6

So $-\frac{3}{2}$ is a zero with multiplicity 2.

The other zeros are found with the quadratic formula

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-1\pm\sqrt{1^2-4(1)(1)}}{2(1)}=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$

We have now determined that the four zeros of the original function f are:

$-\frac{3}{2}$ with multiplicity 2, and the two complex conjugates $\frac{-1+\sqrt{3}i}{2}$, and $\frac{-1-\sqrt{3}i}{2}$

We may now conclude that the graph of f will intersect the x-axis at $-\frac{3}{2}$ but will not cross the axis. It has no other x-intercepts. This permits us to label the one x-intercept and together with the information in Fig. 1 we may produce the graph shown in Fig. 6.

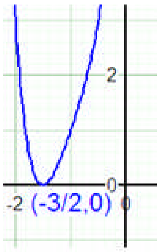

Fig. 6