Constructing a Mathematical Model

From a Standard Formula

In all application problems the first goal is to translate the verbal statement of the problem into a mathematical model of the problem. When such a model is obtained, further discussion of the problem is detached from the language of the application. All imprecision of language and all peculiarities of the application has then been removed and analysis may proceed unimpeded.

In many simple application problems the model is provided by a standard formula. For example the standard formula d = rt expresses the relation between distance, rate, and time as these three quantities relate to rectilinear motion. That formula provides the model for each question about rectilinear motion.
In such applications, the model is obtained by substituting the known quantities into the formula. The solution to the problem is then obtained by solving for the unknown quantity.

In the following examples, three possible questions are discussed for each of two formula. Observe the similarities in the two contexts.

Context	**Rectilinear Motion**	**Percent**
Formula	**d = rt**	**A = PB**
Question Type	**Given r and t, find d.**	**Given P and B, find A.**
Sample Question	If a car travels at 50 mph, how far does it travel in 5 hours?	What is 40% of 60?
Model	d = (50)(5)	A = (40%)(60) = (0.4)(60)
Solution	d = (50)(5) = 250	A = 24
Conclusion	At 50 mph a car travels 250 miles in 5 hours.	24 is 40% of 60
Question Type	**Given d and r, find t.**	**Given A and P, find B.**
Sample Question	How long does it take to travel 250 miles at 50 mph ?	24 is 40% of what number?
Model	250 = 50t	24 = (40%)B = (0.4)B
Solution	t = 5	B = (24/0.4) = 60
Conclusion	At 50 mph a car travels 250 miles in 5 hours.	24 is 40% of 60
Question Type	**Given d and t, find r.**	**Given A and B, find P.**
Sample Question	If a car travels 250 miles in 5 hours, what is its speed?	24 is what percent of 60?
Model	250 = r(5) = 5r	24 = P(60)
Solution	r = 50	P = (24/60) = (0.4) = 40%
Conclusion	At 50 mph a car travels 250 miles in 5 hours.	24 is 40% of 60

Notice the conclusion is the same in each of the three types of questions. That is because each of the three types of questions really ask only for the relation between the three quantities in the formulas.

An Alternate Method

One method for obtaining a mathematical model (equation) for a "real world" problem is to **find some quantity in the problem which may be expressed in two different ways**. Because the two representations are for the same quantity, they must be equal. That observation yields the equation to be solved.

Consider the following examples.

Problem:

A red car and a blue car leave the same starting point at the same time and travel in the same direction. The red car travels at 40 mph while the blue car travels at 55 mph. How long after their departure will the two cars be separated by 5 miles?

Solution:

For each car the distance traveled during any given time may be expressed by using the common formula $d = rt$ for distance. At any time t after their departure from the starting point, the red car has traveled 40t miles and the blue car has traveled 55t miles. **The distance separating the two cars at any given time t may be expressed as 55t – 40t.**

At the time of interest to us in this problem **the distance separating the two cars may be expressed as 5.**

We now have two expressions representing the distance separating the two cars. Those two expressions must be equal.

$55t - 40t = 5$ is the model for this problem.

$55t - 40t = 5$ is the equation which must be solved.

The solution to this equation is easily determined to be 1/3 hour or 20 minutes.

Problem:

How many gallons of a 3% salt solution must be mixed with 50 gallons of a 7% solution to obtain a 5% solution?

Solution:

Let x be the number of gallons of the 3% solution to be added to the 50 gallons of 7% solution.

Note that the result will be (x + 50) gallons. The **amount of salt** in this (x + 50) gallons **is** 5% of (x + 50) or **(.05)(x + 50).** The amount of salt contributed by the two components of the mixture will be:

3% of x gallons or (.03)x plus 7% of 50 gallons or (.07)(50).

The **amount of salt** contributed by the two components **is (.03)x + (.07)(50)**.

We now have two expressions representing the amount of salt in the mixture. Those two expressions must be equal.

$(.05)(x + 50) = (.03)x + (.07)(50)$ is the model for this problem.

$(.05)(x + 50) = (.03)x + (.07)(50)$ is the equation which must be solved.

Begin by multiplying both sides of the equation by 100 to get rid of the decimals

$(.05)(x + 50) = (.03)x + (.07)(50) \Leftrightarrow 5(x + 50) = 3x + (7)(50) \Leftrightarrow 5x + 250 = 3x + 350 \Leftrightarrow 2x = 100 \Leftrightarrow x = 50$

Conclusion:

50 gallons of 3% solution must be added to the 50 gallons of 7% solution to obtain 100 gallons of 5% solution.

Verification:

50 gallons of 3% solution contains 1.5 gallons of salt

50 gallons of 7% solution contains 3.5 gallons of salt

If these two solutions are mixed we obtain 100 gallons containing 5 gallons of salt.

This is clearly a 5% solution. (5 is what percent of 100?).