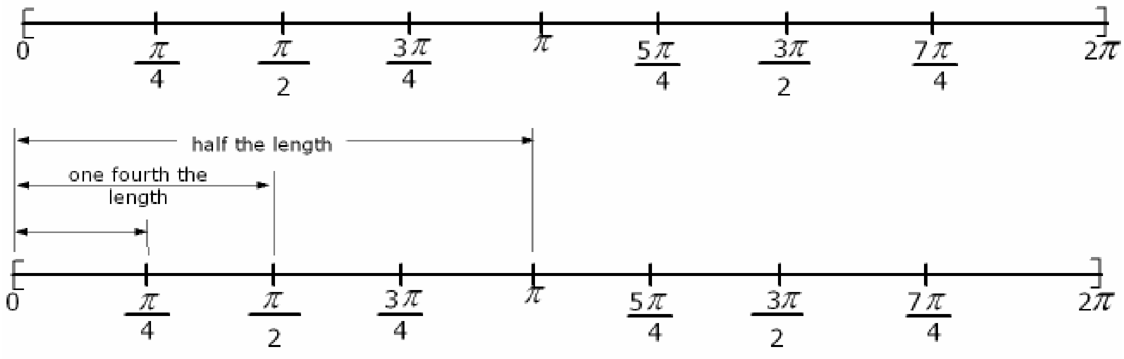
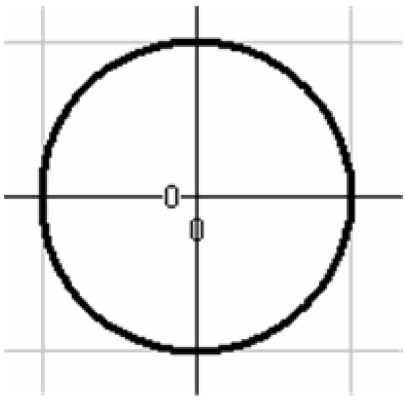
Circular Functions

We will now consider the functions whose names are sn, and cs, whose domain is the closed interval [0, 2π] and whose range is the closed interval [-1, 1]. These and four other functions are called circular functions or more traditionally Trigonometric functions. **This is a simplified (and shifted) version of the sin and cos functions which will be adjusted and extended early in any Trigonometry course.**

Observe the length of this domain is 2π. Here is a picture of the domain of the function named sn. A few points which we will use later in the discussion are marked.



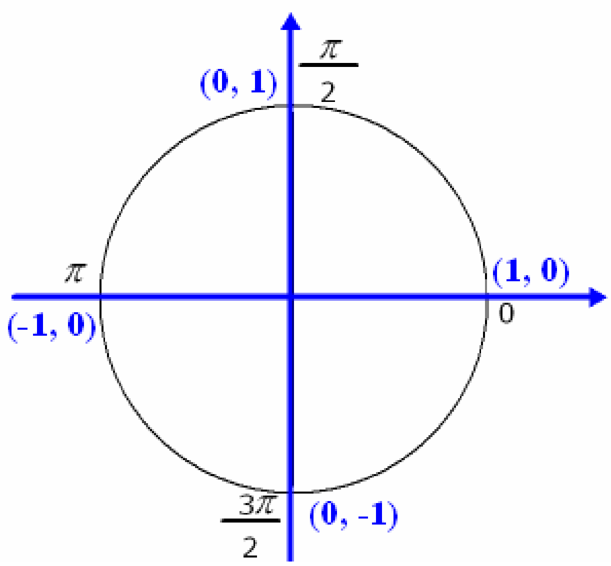


The rule for this function will not be described with an equation but will instead be described in terms of the coordinates of points on the unit circle.

Recall that the unit circle is the circle with radius 1 whose center is at the origin of the Cartesian coordinate system and is described by the equation $x^2 + y^2 = 1$.

Also recall that the radius of a circle is given by the formula C = 2πr. In the case of the unit circle, the circumference is 2π. This circumference is exactly the same length as the domain of the function named sn.

The significance of this comparison is that for any real number in the domain of the function named sn, there is a corresponding point on the unit circle. The converse is also true, for every point on the circumference of the unit circle there is a real number in the domain of the function.



On the unit circle the point (1, 0) is always considered the starting point and distance is always measured on the circumference in the counterclockwise direction.

- The point on the unit circle which corresponds to $\pi/2$ in the domain of sn is the point with coordinates (0, 1).
- The point on the unit circle which corresponds to π in the domain of sn is the point with coordinates (-1, 0).
- The point on the unit circle which corresponds to $3\pi/2$ in the domain of sn is the point with coordinates (-1, -1).
- The point on the unit circle which corresponds to 2π in the domain of sn is the point with coordinates (1, 0).

Whether in the domain of sn or on the circumference of the unit circle, these four points are at the starting point, $\frac{1}{4}$ the total distance, $\frac{1}{2}$ the total distance, and $\frac{3}{4}$ the total distance. We are now ready to provide the rule for the functions named sn and cs.

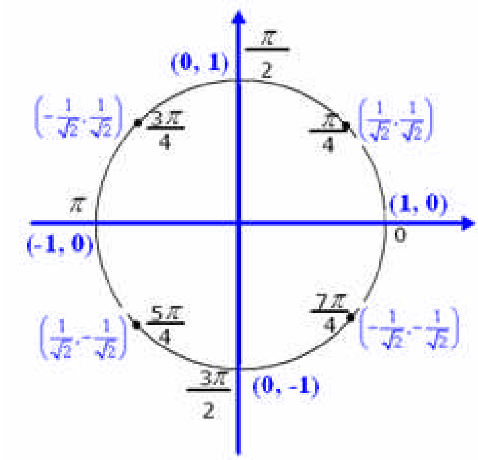
RULE: For any $x \in [0, 2\pi]$, sn(x) is the second coordinate of the corresponding point on the circumference of the unit circle.
RULE: For any $x \in [0, 2\pi]$, cs(x) is the first coordinate of the corresponding point on the circumference of the unit circle.

The above diagram shows that:

$$\text{sn}(0) = 0, \quad \text{sn}\left(\frac{\pi}{2}\right) = 1, \quad \text{sn}(\pi) = 0, \quad \text{sn}\left(\frac{3\pi}{2}\right) = -1, \quad \text{sn}(2\pi) = 0$$

$$\text{cs}(0) = 1, \quad \text{cs}\left(\frac{\pi}{2}\right) = 0, \quad \text{cs}(\pi) = -1, \quad \text{cs}\left(\frac{3\pi}{2}\right) = 0, \quad \text{cs}(2\pi) = 1$$

We will now look at the range values associated with a few other domain elements. In particular we will examine those numbers (domain elements) midway between each pair of the previous four numbers in the domain of sn and cs.

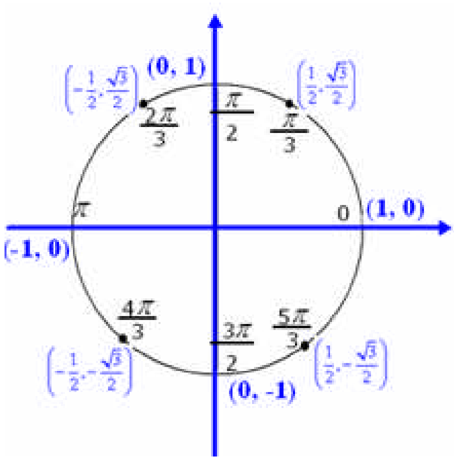


Recall the rules for the functions sn and cs and extract the following range values directly from the picture at the right.

$$\text{sn}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ \text{sn}\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ \text{sn}\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \ \text{sn}\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\text{cs}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ \text{cs}\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \ \text{cs}\left(\frac{5\pi}{4}\right) = \frac{1}{\sqrt{2}}, \ \text{cs}\left(\frac{7\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

The figure at the right shows additional points in the domain of sn and cs with their coordinates on the unit circle. From this diagram and the rules for the two functions we can conclude:



$$\text{sn}\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2},\ \ \text{sn}\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2},\ \ \text{sn}\left(\frac{4\pi}{3}\right)=-\frac{\sqrt{3}}{2},\ \ \text{sn}\left(\frac{5\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$

$$\text{cs}\left(\frac{\pi}{3}\right)=\frac{1}{2},\ \ \text{cs}\left(\frac{2\pi}{3}\right)=-\frac{1}{2},\ \ \text{cs}\left(\frac{4\pi}{3}\right)=-\frac{1}{2},\ \ \text{cs}\left(\frac{5\pi}{3}\right)=\frac{1}{2}$$

In a manner similar to the examples presented in these few examples, the unique range values associated with a domain element may be determined.

Each point in the domain of sn and cs corresponds with a point on the unit circle which in turn corresponds with a set of first and second coordinates which determine the unique range value associated with the domain element.

The two functions sn and cs and four other circular functions are the functions studied in Trigonometry. How these functions relate to angles, triangles, radian measure, etc. will not be discussed here. The purpose here is simply to give an illustration of some functions whose rules are unusual and whose domains and ranges are not all of **R**.