

Important Properties of Polynomial Functions

Definition: Polynomial Function

A polynomial function is a function whose rule may be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

where each a_i is a real number and n is a natural number.

Definition: Degree

The exponent n of the leading term $a_n x^n$ of the polynomial function f , is called the degree of the function f .

Fact: For domain elements far from the origin, the leading term in a polynomial function dominates the entire expression when calculating range elements.

Fact: The graph of a polynomial function is a continuous smooth graph with no sharp corners.

Fact: The graph of a polynomial function f of degree n can have no more than n x -intercepts. The graph “tries” to have exactly n x -intercepts.

Fact: The graph of a polynomial function f of degree n can have no more than $n - 1$ turning points (humps). The graph “tries” to have exactly $n-1$ humps.

Fact: Factor Theorem

A polynomial function f has a factor $(x - k)$ if and only if $f(k) = 0$. The above statement means:

If $(x - k)$ is a factor of the polynomial function f , then $f(k) = 0$.

If $f(k) = 0$ for a polynomial function f , then $(x - k)$ is a factor of f .

Fact: Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$ then, in the interval $[a, b]$ f takes on every value between $f(a)$ and $f(b)$.

Fact: Fundamental Theorem of Algebra

If f is a polynomial of degree n , then f has at least one zero in the complex number system.

Fact: Let f be a polynomial function with real coefficients.

If $a + bi$ with $b \neq 0$ is a zero of f , then the conjugate $a - bi$ is a zero of f .

Fact: Factors of a Polynomial

Every polynomial function of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Fact: Rational Zeros

If f is a polynomial function with integer coefficients then every rational zero has the form $\frac{p}{q}$ such that:

- the only common factor of p and q is 1
- p is a factor of the constant term a_0
- q is a factor of the leading term a_n

Analysis of Polynomial Functions

Question 1:

Consider the function f whose rule is:

$$f(x) = \frac{1}{3}x^3(x-4)^2$$

Analysis:

The function f is a 5th degree polynomial function. Its domain is \mathbf{R} and its range is \mathbf{R} . Its graph will be a continuous smooth curve with no sharp corners and it will try to have five x-intercepts.

Consideration of the leading term $\frac{1}{3}x^5$ leads to the following: $f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$
 $f(x) \longrightarrow -\infty$ as $x \longrightarrow -\infty$

We find the zeros of f by solving the equation resulting from $f(x) = 0$. Therefore we must solve

$$\begin{aligned} \frac{1}{3}x^3(x-4)^2 &= 0 \\ x^3 &= 0 \quad \text{or} \quad (x-4)^2 = 0 \\ x &= 0 \quad \text{or} \quad x = 4 \end{aligned}$$

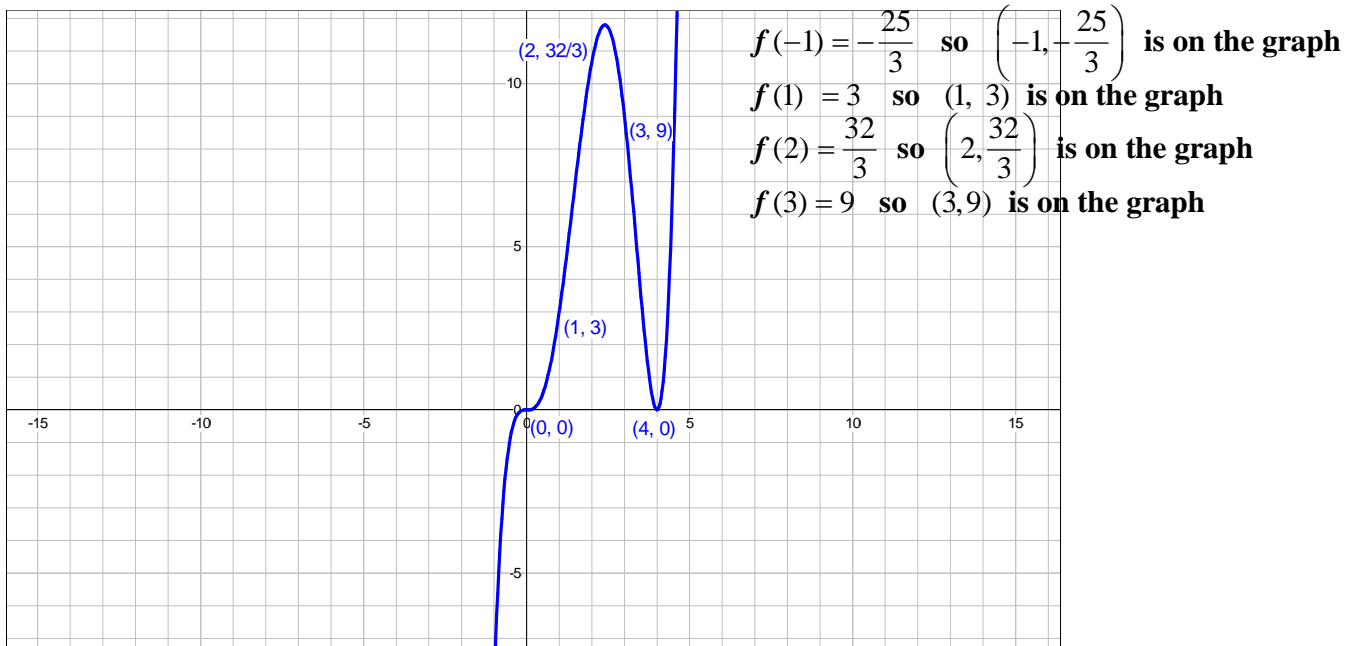
4 is a real zero of multiplicity 2, an even number.

The graph will therefore touch, but not cross, the x-axis at 4.

0 is a real zero of multiplicity 3, an odd number.

The graph will therefore cross the x-axis at $x = 0$.

For this function I have plotted a few additional points to better illustrate the graph of f as produced by a graphing utility and displayed below.. For this purpose I computed:



From the work above we observe

$$f(x) = 0 \text{ if } x \in \{0, 4\}$$

From the graph we can observe

$$f(x) > 0 \text{ if } x \in (0, 4) \cup (4, +\infty)$$

$$f(x) < 0 \text{ if } x \in (-\infty, 0)$$

Question 2:

Analyze the function g whose rule is $f(x) = x^4 - 4x^2$

Analysis:

The function f is a fourth degree polynomial function. Its domain is \mathbf{R} and its range is \mathbf{R} . Its graph will be a continuous smooth curve with no sharp corners and it will try to have four x-intercepts.

Consideration of the leading term x^4 leads to the following: $f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$
 $f(x) \longrightarrow +\infty$ as $x \longrightarrow -\infty$

We find the zeros of f by solving the equation resulting from $f(x) = 0$. Therefore we must solve

$$0 = x^4 - 4x^2$$

$$0 = x^4 - 4x^2 = x^2(x^2 - 4) = x^2(x - 2)(x + 2)$$

The solutions are -2, 0 and 2.

The zeros are all real numbers and therefore correspond to x-intercepts.

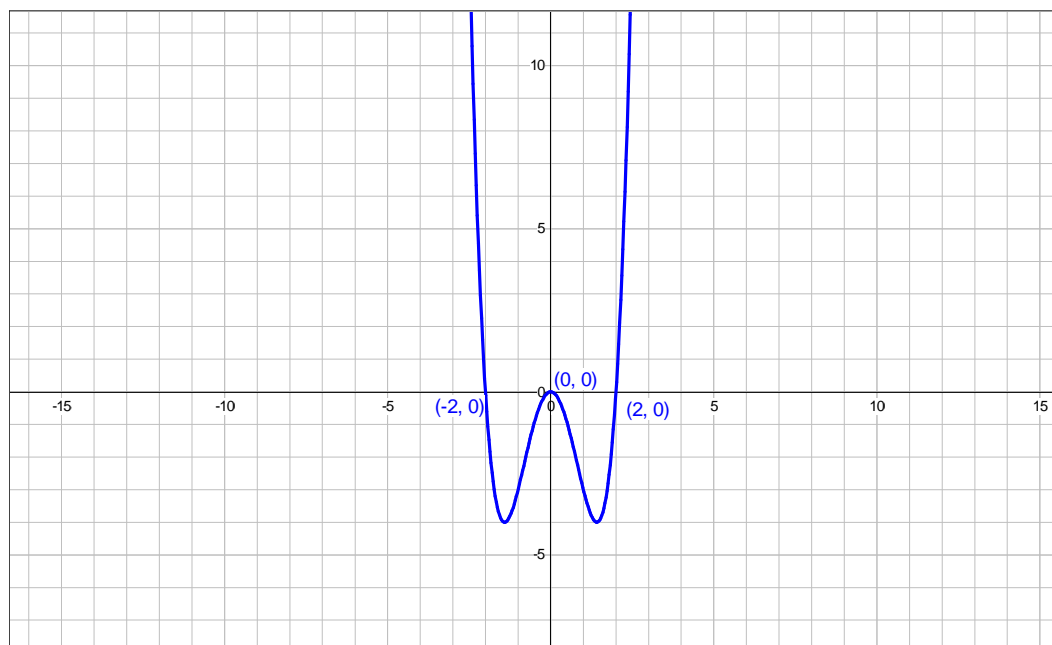
0 is a zero of multiplicity 2, an even number.

Therefore the graph touches, but does not cross, the x-axis at 0.

-2 and 2 each have multiplicity 1, an odd number.

Therefore the graph will cross the x-axis at -2 and 2.

The graph of f as generated by a graphing utility is shown below.



The above work shows that”

$$f(x) = 0 \text{ if } x \in \{-2, 0, 2\}$$

From the graph we can deduce:

$$f(x) > 0 \text{ if } x \in (-\infty, -2) \cup (2, +\infty)$$

$$f(x) < 0 \text{ if } x \in (-2, 0) \cup (0, 2)$$

Polynomial Function – Analysis and Graphing

Problem: Analyze and Sketch the Graph of the function f whose rule is

$$f(x) = 4x^4 + 16x^3 + 25x^2 + 21x + 9$$

Analysis:

The function f is a polynomial function. Therefore its graph will be a smooth continuous curve with no gaps or sharp corners.

The degree of f is four. Therefore its graph tries to cross the x -axis four times and tries to exhibit three turning points (humps).

The degree of f is even. Therefore its graph need not have an x -intercept.

Because f is a polynomial function the leading term will dominate when x is far from the origin. Therefore

$$\text{as } x \rightarrow \infty, f(x) \rightarrow \infty \text{ and as } x \rightarrow -\infty, f(x) \rightarrow \infty$$

We now have enough information to expect the graph of f to look like some variation of the graph in Fig. 1. However, we must recognize that the number of x -intercepts and the number of turning points might be very different than shown in Fig. 1.

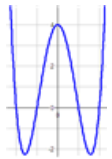


Fig. 1

To determine the x -intercepts of the graph of f we must find the zeros of f .

To find the zeros of f (as with all functions) we must solve the equation resulting from $f(x) = 0$.

In this case we must solve the equation $0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9$

We cannot solve this equation directly, so we turn to the rational zeros theorem to find the possible rational zeros of the function f .

If $\frac{p}{q}$ is a rational zero of the function f , then p is a divisor of the constant term 9 and q is a divisor of the

leading coefficient 4. So $p \in N = \{\pm 1, \pm 3, \pm 9\}$ and $q \in D = \{\pm 1, \pm 2, \pm 4\}$ and then $\frac{p}{q}$ must be in the

set of all fractions which may be constructed by selecting numerators from N and selecting denominators

from D . Therefore $\frac{p}{q} \in \left\{ \pm 1, \pm 3, \pm 9, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}, \pm \frac{1}{4}, \pm \frac{3}{4}, \pm \frac{9}{4} \right\} = K$. Thus if the function f has a

rational zero (a rational x -intercept) then it must be one of the above 18 rational numbers in the set labeled K .

To avoid testing each of these, it is wise at this point to use a graphing utility such as Omnigraph or a TI 83 to produce a graph of f as shown in Fig. 2. The graph in Fig. 2 makes it clear that the only reasonable options for rational zeros must be in the interval $(-2, -1)$. The only element of the

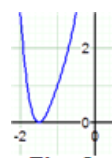


Fig. 2

set K in this interval is the number $-\frac{3}{2}$. We must now test $-\frac{3}{2}$ to see if it is a zero of the

function f . That is we must determine if $f\left(-\frac{3}{2}\right) = 0$. We can avoid substituting $-\frac{3}{2}$ into the rule for f

by recalling that the following statements are equivalent.

1) $-\frac{3}{2}$ is a zero of f

2) $x - \left(-\frac{3}{2}\right) = x + \frac{3}{2}$ is a factor of $4x^4 + 16x^3 + 25x^2 + 21x + 9$

But even the prospect of performing this division with a fraction in the divisor is not appealing. However, if we note that

$x + \frac{3}{2}$ is a factor of $4x^4 + 16x^3 + 25x^2 + 21x + 9$ if and only if

$2x + 3$ is a factor of $4x^4 + 16x^3 + 25x^2 + 21x + 9$

So we perform the later division as shown in Fig. 3 to obtain a quotient of

$2x^3 + 5x^2 + 5x + 3$ and a remainder of 0. This shows that

$$0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9 = (2x + 3)(2x^3 + 5x^2 + 5x + 3)$$

And now The Zero Factor Property implies

$$2x + 3 = 0 \text{ OR } 2x^3 + 5x^2 + 5x + 3 = 0$$

$$x = -\frac{3}{2} \text{ OR } 2x^3 + 5x^2 + 5x + 3 = 0$$

Thus $-\frac{3}{2}$ is a zero of f. To determine the other zeros of f we must solve the equation $2x^3 + 5x^2 + 5x + 3 = 0$.

Note that is the same as finding the zeros of the function g whose rule is

$$g(x) = 2x^3 + 5x^2 + 5x + 3.$$

The graph of g is shown in red superimposed over the graph of f in Fig. 4.

It appears from Fig. 4 that $-\frac{3}{2}$ is a zero of g. So it appears that

$2x + 3$ is a factor of $2x^3 + 5x^2 + 5x + 3$.

This fact is verified in the division shown in Fig.5.

This division yields the following

$$0 = 4x^4 + 16x^3 + 25x^2 + 21x + 9 = (2x + 3)(2x^3 + 5x^2 + 5x + 3) = (2x + 3)^2(x^2 + x + 1)$$

And again we invoke The Zero Factor Property to conclude that

$$(2x + 3)^2 = 0 \text{ OR } x^2 + x + 1 = 0$$

$$x = -\frac{3}{2} \text{ OR } x^2 + x + 1 = 0$$

So $-\frac{3}{2}$ is a zero with multiplicity 2.

The other zeros are found with the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

We have now determined that the four zeros of the original function f are:

$$-\frac{3}{2} \text{ with multiplicity 2, and the two complex conjugates } \frac{-1 + \sqrt{3}i}{2}, \text{ and } \frac{-1 - \sqrt{3}i}{2}$$

We may now conclude that the graph of f will intersect the x-axis at $-\frac{3}{2}$ but will not cross the

axis. It has no other x-intercepts. This permits us to label the one x-intercept and together with the information in Fig. 1 we may produce the graph shown in Fig. 6.

$$\begin{array}{r} 2x^3 + 5x^2 + 5x + 3 \\ 2x + 3 \overline{) 4x^4 + 16x^3 + 25x^2 + 21x + 9} \\ \underline{4x^4 + 6x^3} \\ 10x^3 + 25x^2 + 21x + 9 \\ \underline{10x^3 + 15x^2} \\ 10x^2 + 21x + 9 \\ \underline{10x^2 + 15x} \\ 6x + 9 \\ \underline{6x + 9} \\ 0 \end{array}$$

Fig. 3

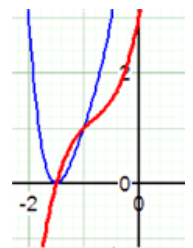


Fig. 4

$$\begin{array}{r} x^2 + x + 1 \\ 2x + 3 \overline{) 2x^3 + 5x^2 + 5x + 3} \\ \underline{2x^3 + 3x^2} \\ 2x^2 + 5x + 3 \\ \underline{2x^2 + 3x} \\ 2x + 3 \\ \underline{2x + 3} \\ 0 \end{array}$$

Fig. 6

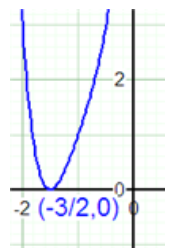


Fig. 6

Exercise Polynomial

Question:

Analyze the function f whose rule is $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

It is known that $-3 + \sqrt{2}i$ is a zero of f .

Solution:

The function f is a 4th degree polynomial function. Its domain and range is all real numbers \mathbf{R} . Its graph will be a continuous smooth curve with no sharp corners and it will try to have four x -intercepts.

Consideration of the leading term x^4 leads to the following: $f(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$
 $f(x) \longrightarrow +\infty$ as $x \longrightarrow -\infty$

Complex zeros occur in conjugate pairs, so both $-3 + \sqrt{2}i$ and $-3 - \sqrt{2}i$ are zeros of f .

Then both $[x - (-3 + \sqrt{2}i)]$ and $[x - (-3 - \sqrt{2}i)]$ are factors.

Therefore their product is a factor of the function f .

Compute the product to find:

$$\begin{aligned} [x - (-3 + \sqrt{2}i)][x - (-3 - \sqrt{2}i)] &= x^2 - (-3 - \sqrt{2}i)x - (-3 + \sqrt{2}i)x + (-3 + \sqrt{2}i)(-3 - \sqrt{2}i) \\ &= x^2 + 3x + \sqrt{2}ix + 3x - \sqrt{2}ix + [9 + 3\sqrt{2}i - 3\sqrt{2}i - \sqrt{2}i\sqrt{2}i] \\ &= x^2 + 6x + [9 - (\sqrt{2}i)^2] = x^2 + 6x + 11 \end{aligned}$$

Therefore $x^2 + 6x + 11$ is a factor of $x^4 + 3x^3 - 5x^2 - 21x + 22$

Ordinary long division produces a quotient of $x^2 - 3x + 2$

$$\begin{array}{r} x^2 - 3x + 2 \\ x^2 + 6x + 11 \overline{) x^4 + 3x^3 - 5x^2 - 21x + 22} \\ \underline{x^4 + 6x^3 + 11x^2} \\ -3x^3 - 16x^2 - 21x + 22 \\ \underline{-3x^3 - 18x^2 - 33x} \\ 2x^2 + 12x + 22 \\ \underline{2x^2 + 12x + 22} \\ 0 \end{array}$$

Therefore

$$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22 = (x^2 + 6x + 11)(x^2 - 3x + 2) = (x^2 + 6x + 11)(x - 2)(x - 1)$$

We find the zeros of f by solving the equation resulting from $f(x) = 0$.

Therefore we must solve the equation $(x^2 + 6x + 11)(x - 2)(x - 1) = 0$.

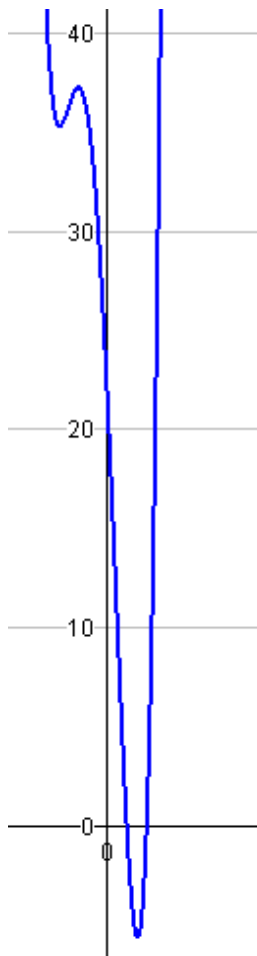
The four solutions of this equation and hence the four zeros of the polynomial function f are

$1, 2, -3 + \sqrt{2}i$ and $-3 - \sqrt{2}i$.

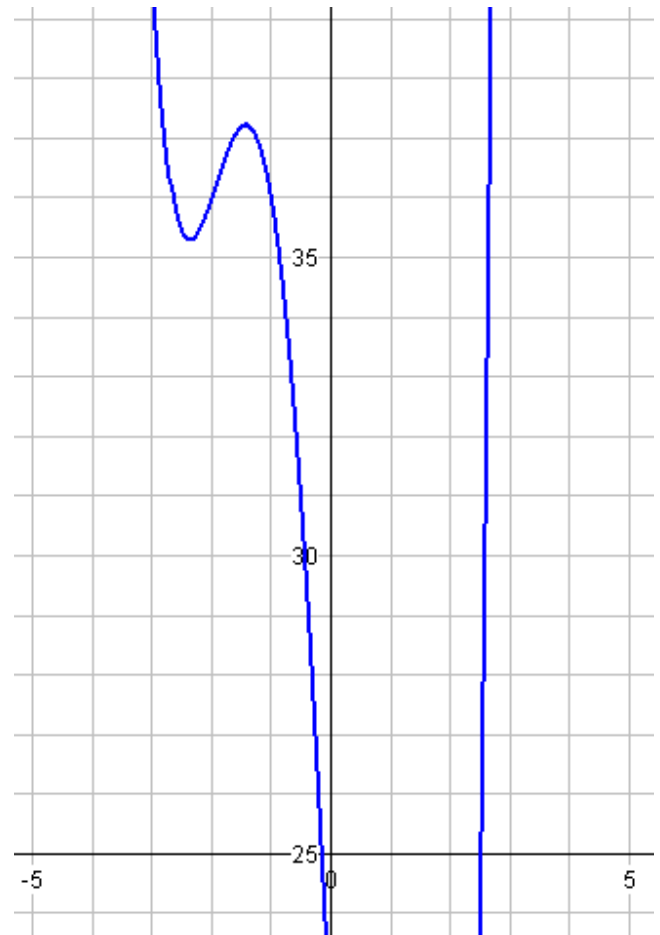
Two of these zeros are real numbers and therefore represent x -intercepts. Both have odd multiplicity so the graph crosses the x -axis at the intercepts.

We might observe that the graph crosses the y -axis at 22.

The first graph shows the shape of the graph. The other two are blow-ups to show certain features of the graph. Note the scale on the axis when viewing these three graphs.

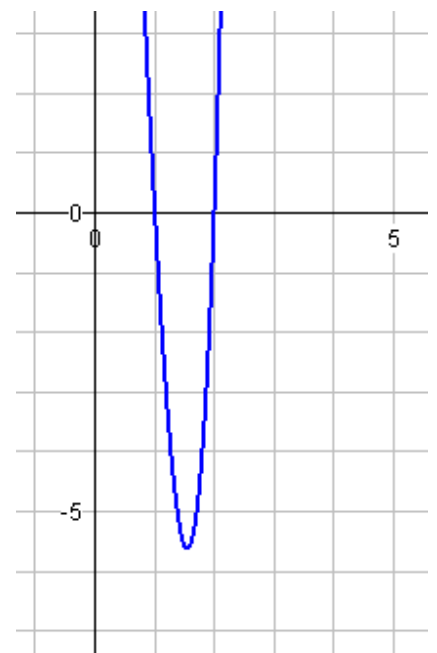


Notice that the graph “tries” to cross the x-axis four times, but the attempt between 0 and -5 just does not drop down far enough.



The graph of a fourth degree polynomial “tries” to have three humps (turning points) and in fact this graph does have three such turning points.

Also observe that the graph is smooth, continuous, with no sharp corners, exactly as one would expect from a polynomial function.



If you did not observe the quadratic factor but used the Rational

Zeros Test from the beginning, then you would observe that if $\frac{p}{q}$ is a rational zero then p is a factor of the constant term 22 and 1 is a factor of the leading coefficient 1.

Therefore $p \in \{\pm 1, \pm 2, \pm 11, \pm 22\}$ and $q \in \{\pm 1\}$

so $\frac{p}{q} \in \{\pm 1, \pm 2, \pm 11, \pm 22\}$

Test 1: $f(1) = 1 + 3 - 5 - 21 + 22 = 0$

Therefore 1 is a zero of f and $x - 1$ is a factor. Division yields a quotient of $x^3 + 4x^2 - x - 22$.

Then $f(x) = (x - 1)(x^3 + 4x^2 - x - 22)$

Now we need only look for zeros in $x^3 + 4x^2 - x - 22$. But we must check 1 again because it might be a zero with multiplicity greater than 1. To facilitate notation define g to be the function whose rule is $g(x) = x^3 + 4x^2 - x - 22$

Test 1 again: $g(1) = 1 + 4 - 1 - 22 < 0$. So 1 is not a zero of g. This implies 1 is a real zero of the function f and it has multiplicity 1, an odd number. So the graph will cross the x-axis at 1.

Test -1: $g(-1) = (-1)^3 + 4(-1)^2 - (-1) - 22 < 0$. Therefore -1 is not a zero of the function f.

Test 2: $g(2) = (2)^3 + 4(2)^2 - (2) - 22 = 0$. Therefore 2 is a zero of both g and f and $x - 2$ is a factor of both g and f. Normal division of g by $x - 2$ yields a quotient of $x^2 + 6x + 11$.

Then $g(x) = (x^3 + 4x^2 - x - 22) = (x - 2)(x^2 + 6x + 11)$

From which we get

$$f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22 = (x - 1)g(x) = (x - 1)(x - 2)(x^2 + 6x + 11)$$

At this point one should observe that we have found all possible real zeros because this function has exactly four zeros and two of them must be complex, because complex zeros occur in conjugate pairs.

All other analysis should be as presented on Pages 1 and 2.

Polynomial Analysis Illustrations

Question

Analyze the function whose rule is $f(x) = 2x^3 + 3x^2 - 1$

Analysis: $f(x) = 2x^3 + 3x^2 - 1$ is the rule for a third degree polynomial named f. Therefore the graph of f will be a smooth continuous curve with no sharp corners.

The graph will try to cross the x-axis three times.

The graph can have no more than three x-intercepts and because the degree is an odd number, the graph will have at least one x-intercept.

As $x \longrightarrow \infty, f(x) \longrightarrow \infty$

As $x \longrightarrow -\infty, f(x) \longrightarrow -\infty$

The extremes of the graph look like



If $\frac{p}{q}$ is a zero of f, then

$$p \in \{\pm 1\} \text{ and } q \in \{\pm 1, \pm 2\}$$

$$\text{Therefore } \frac{p}{q} \in \left\{ \pm 1, \pm \frac{1}{2} \right\}$$

Test 1: $f(1) = 2 + 3 - 1 > 0$ so 1 is not a zero of f.

Test -1: $f(-1) = 2(-1)^3 + 3(-1)^2 - 1 = -2 + 3 - 1 = 0$

Therefore -1 is a zero of the function f and

$x+1$ is a factor of the polynomial $2x^3 + 3x^2 - 1$

Use ordinary long division of polynomials to find the quotient when $2x^3 + 3x^2 - 1$ is divided by $x + 1$.

$$\begin{array}{r} 2x^2 + x - 1 \\ x+1 \overline{) 2x^3 + 3x^2 - 1} \\ \underline{2x^3 + 2x^2} \\ x^2 - 1 \\ \underline{x^2 + x} \\ -x - 1 \\ \underline{-x - 1} \\ 0 \end{array}$$

$$\begin{aligned} \text{Therefore } f(x) &= 2x^3 + 3x^2 - 1 = (x+1)(2x^2 + x - 1) \\ &= (x+1)(2x-1)(x+1) = (x+1)^2(2x-1) \end{aligned}$$

Now it is clear (from the Zero Factor Property) that

the zeros of f are -1, and $\frac{1}{2}$.

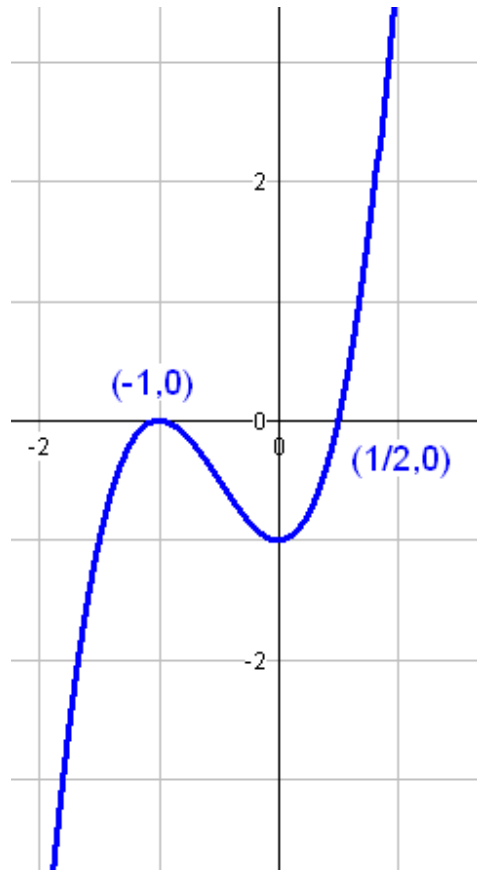
Observe that -1 is a zero with multiplicity 2, an even number.

The graph of f will intersect but not cross the x-axis at -1.

Observe that $\frac{1}{2}$ is a zero of multiplicity 1, an odd number.

The graph of f will cross the x-axis at $\frac{1}{2}$

The graph of f is shown below. Observe that although this graph is computer generated, the same graph will be obtained if a graph is sketched which conforms to the information we derived from the earlier analysis.



One of the most important reasons for sketching the graph of a function is to clearly show the answers to the following three very important questions.

- i) Where is $f(x) = 0$?
- ii) Where is $f(x) > 0$?
- iii) Where is $f(x) < 0$?

From the above graph it is easy to see the answers to those very important questions.

$$f(x) = 0 \text{ if } x \in \left\{ -1, \frac{1}{2} \right\}$$

$$f(x) > 0 \text{ if } x \in \left(\frac{1}{2}, +\infty \right)$$

$$f(x) < 0 \text{ if } x \in (-\infty, -1) \cup \left(-1, \frac{1}{2} \right)$$

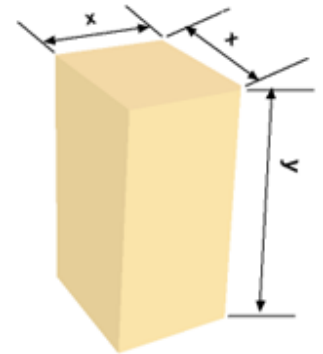
Notice the Law of Trichotomy at work.

Question

A rectangular box to be shipped by the U.S. Postal service can have a maximum combined length and girth (perimeter of a cross section) of 108 inches. Find the rule for a function V which models the volume of boxes which have combined length and girth equal to 108 inches and whose cross section is a square. The domain of V should be the set of numbers which can be the dimension of the square cross section. Analyze the function V as completely as possible. Interpret the analysis of the function in terms of shipping boxes.

Creating the Model:

The picture at the right is a geometric representation of the desired box. The requirement that the cross section be a square is reflected in the labeling of the picture by making x equal to both dimensions of the cross section. The terminology “combined length and girth” means the sum of girth and length. The sum of girth and length can be represented two ways which will give us a solvable equation.



Note that the girth is $4x$. The length is y .

The sum of girth and length is $4x + y$ and on the other hand the sum of the girth and length is 108. This gives the equation $4x + y = 108$.

Notice that $y = 108 - 4x$.

The volume of the box is given by $\text{Volume} = x^2y = x^2(108 - 4x)$

We are now in a position to create the desired functional model.

The domain of the function is all values of x which make sense. Clearly x cannot be 0 or negative so x must be greater than 0 Algebraically we write this as $x \geq 0$. Similarly y cannot be 0 or negative $0 \leq y = 108 - 4x$ If we solve $0 \leq 108 - 4x$ we get $4x \leq 108$

and finally we find $x \leq 27$

The domain of the desired function V is the interval $(0, 27)$

The rule for the desired function V is $V(x) = x^2(108 - 4x)$

Analysis:

The function V is a third degree polynomial function whose leading term is $-4x^3$.

Therefore its graph is a smooth continuous curve with no sharp corners which tries to cross the x -axis in three places.

If we temporarily ignore the domain of V we observe that because its degree is odd, the graph will have at least one x -intercept.

We can also observe behavior far from the origin to determine the orientation of graph of this cubic.

$$\text{as } x \longrightarrow \infty, f(x) \longrightarrow -\infty$$

$$\text{as } x \longrightarrow -\infty, f(x) \longrightarrow \infty$$

The zeros of the function V are found by solving the equation resulting from $V(x) = 0$. In this case we must solve the equation $x^2(108 - 4x) = 0$. The zero Factor Property yields two easily solved equations and we get solutions 0 and 27. Because both of these are outside the domain of f the function V has no zeros and therefore its graph has no x -intercepts. Notice how the hunt for zeros led to solutions 0 and 27 which agrees with our earlier analysis while determining the domain of V .



Graph of V with domain \mathbf{R} and rule
 $V(x) = x^2(108 - 4x)$



Graph of V with domain $(0, 27)$ and rule
 $V(x) = x^2(108 - 4x)$

Interpretation:

Observe that the graph of $V(x) = x^2(108 - 4x)$ with domain \mathbf{R} shows clearly that $V(x) > 0$ if $x \in (0, 27)$

Some computations will help to understand what this function V and its graph represent. If the dimension of the cross section is 5, the volume is the unique range element associated with the domain element 5, so the volume is $V(5)$ and of course we have a rule for V which enables us to compute $V(5) = 5^2(108 - 4(5)) = 25(108 - 20) = 2200$

Similarly when the dimension of the cross section is 10, 15, 18, 20, and 25 respectively, the function computes the volume for us:

$$V(10) = 10^2(108 - 4(10)) = 100(68) = 6,800$$

$$V(15) = 15^2(108 - 4(15)) = 225(48) = 10,800$$

$$V(18) = 18^2(108 - 4(18)) = 324(36) = 11,664$$

$$V(20) = 20^2(108 - 4(20)) = 400(28) = 11,200$$

$$V(25) = 25^2(108 - 4(25)) = 625(8) = 5,000$$

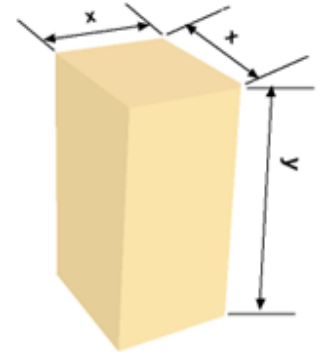
It appears from the graph and the computations shown in the EXCEL table at the right, that the maximum volume will be attained if the dimension of the square cross section is 18 inches.

Determination, with certainty, of the domain element which yields the maximum volume is a calculus topic. In that calculus analysis it is necessary to find the zeros of a function V' whose rule is $V'(x) = -12x^2 + 216x = -12x(x - 18)$ whose zeros are clearly 0 and 18. The number 0 is not in the domain of the function but 18 is in the domain and the maximum range value will be $V(18)$.

X	V(X)
17.5	11637.5000
17.55	11642.4945
17.65	11650.9415
17.7	11654.3880
17.75	11657.3125
17.8	11659.7120
17.85	11661.5835
17.9	11662.9240
17.95	11663.7305
17.995	11663.9973
18	11664.0000
18.005	11663.9973
18.05	11663.7295
18.1	11662.9160
18.15	11661.5565
18.2	11659.6480
18.25	11657.1875
18.3	11654.1720
18.35	11650.5985
18.4	11646.4640
18.45	11641.7655
18.5	11636.5000

Question

A rectangular box to be shipped by the AJAX Delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches. Find the rule for a function V which models the volume of boxes which have combined length and girth equal to 120 inches and whose cross section is a square. The domain of V should be the set of numbers which can be the dimension of the square cross section. Analyze the function V as completely as possible. Interpret the analysis of the function in terms of shipping boxes. Determine the value of x for which the volume is 13,500. Determine the value of x for which the volume is maximum.



Creating the Model:

The picture at the right is a geometric representation of the desired box. The requirement that the cross section be a square is reflected in the labeling of the picture by making x equal to both dimensions of the cross section. The terminology “combined length and girth” means the sum of girth and length. The sum of girth and length can be represented two ways which will give us a solvable equation.

Note that the girth is $4x$. The length is y .

The sum of girth and length is $4x + y$ and on the other hand the sum of the girth and length is 120. This gives the equation $4x + y = 120$. Notice that $y = 120 - 4x$.

The volume of the box is given by $\text{Volume} = x^2 y = x^2 (120 - 4x) = 4x^2 (30 - x)$

We are now in a position to create the desired functional model.

The domain of the function is all values of x which make sense. Clearly x cannot be 0 or negative so x must be greater than 0. Algebraically we write this as $x \geq 0$. Similarly y cannot be 0 or negative $0 \leq y = 120 - 4x$ If we solve $0 \leq 120 - 4x$ we get $4x \leq 120$

and finally we find $x \leq 30$

The domain of the desired function V is the interval $(0, 30)$

The rule for the desired function V is $V(x) = x^2 (120 - 4x)$

Analysis:

The function V is a third degree polynomial function whose leading term is $-4x^3$.

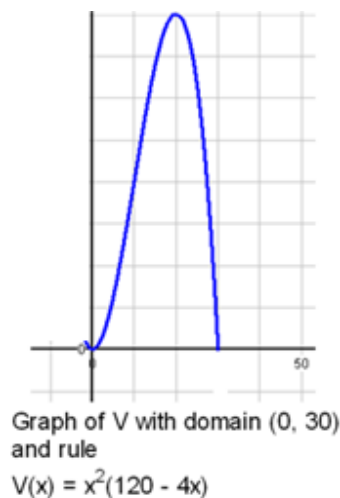
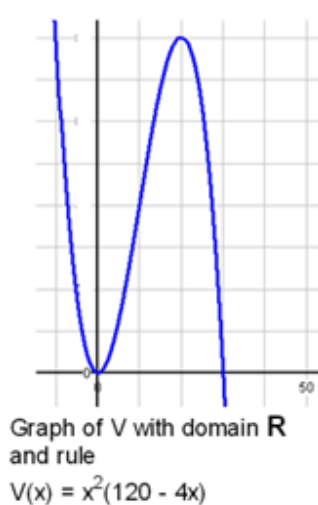
Therefore its graph is a smooth continuous curve with no sharp corners which tries to cross the x -axis in three places.

If we temporarily ignore the domain of V we observe that because its degree is odd, the graph will have at least one x -intercept.

We can also observe behavior far from the origin to $\text{as } x \longrightarrow \infty, f(x) \longrightarrow -\infty$

determine the orientation of graph of this cubic. $\text{as } x \longrightarrow -\infty, f(x) \longrightarrow \infty$

The zeros of the function V are found by solving the equation resulting from $V(x) = 0$. In this case we must solve the equation $x^2 (120 - 4x) = 0$. The zero Factor Property yields two easily solved equations and we get solutions 0 and 30. Because both of these are outside the domain of f the function V has no zeros and therefore its graph has no x -intercepts. Notice how the hunt for zeros led to solutions 0 and 30 which agrees with our earlier analysis while determining the domain of V .



Interpretation:

Observe that the graph of $V(x) = x^2(120 - 4x)$ with domain \mathbf{R} shows clearly that $V(x) > 0$ if $x \in (0, 30)$

Some computations will help to understand what this function V and its graph represent.

If the dimension of the cross section is 5, the volume is the unique range element associated with the domain element 5, so the volume is $V(5)$ and of course we have a rule for V which enables us to compute $V(5) = 5^2(120 - 4(5)) = 25(120 - 20) = 2500$

20 Similarly when the dimension of the cross section is 10, 15, 20, and 25 respectively, the function computes the volume for us:

$$V(10) = 10^2(120 - 4(10)) = 100(68) = 6,800$$

$$V(15) = 15^2(120 - 4(15)) = 225(60) = 13,500$$

$$V(20) = 20^2(120 - 4(20)) = 400(40) = 16,000$$

$$V(25) = 25^2(120 - 4(25)) = 625(20) = 12,500$$

It appears from the graph and the computations shown in the EXCEL table at the right, that the maximum volume will be attained if the dimension of the square cross section is 20 inches.

Determination, with certainty, of the domain element which yields the maximum volume is a calculus topic. In that calculus analysis it is necessary to find the zeros of a function

V' whose rule is $V'(x) = -12x^2 + 240x = -12x(x - 20)$ whose zeros are clearly 0 and 20. The number 0 is not in the domain of the function but 20 is in the domain and the maximum range value will be $V(20)$.

X	V(X)
19.5	15970.5000
19.55	15976.0645
19.65	15985.4715
19.7	15989.3080
19.75	15992.5625
19.8	15995.2320
19.85	15997.3135
19.9	15998.8040
19.95	15999.7005
19.995	15999.9970
20	16000.0000
20.005	15999.9970
20.05	15999.6995
20.1	15998.7960
20.15	15997.2865
20.2	15995.1680
20.25	15992.4375
20.3	15989.0920
20.35	15985.1285
20.4	15980.5440
20.45	15975.3355
20.5	15969.5000

Find values of x for which the volume is 13,500.

In terms of the function V, we want to find a domain element t such that $V(t) = 13,500$.

We must therefore solve the equation resulting from $V(t) = 13,500$. In this case we must solve the equation $x^2(120 - 4x) = 13500$ which is equivalent to $-4x^3 + 120x^2 - 13500 = 0$.

This is equivalent to finding the zeros of the polynomial function $f(x) = -4x^3 + 120x^2 - 13500$. Use of the Rational Zeros Property, some testing and a little luck determines that 15 is a zero of the function f. Therefore $x - 15$ is a factor of the polynomial $-4x^3 + 120x^2 - 13500$. Ordinary long division determines the quotient to be $x^2 - 15x - 225$ so that $f(x) = -4x^3 + 120x^2 - 13500 = (x - 15)(x^2 - 15x - 225)$.

According to the Zero Factor Property, the other two zeros of f can be found by solving the equation $x^2 - 15x - 225 = 0$ using the Quadratic Formula.

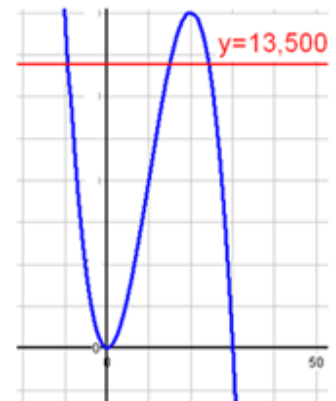
The computations are:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

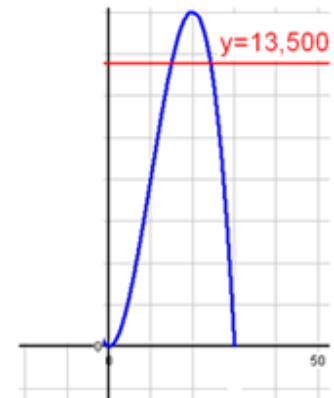
$$= \frac{15 \pm \sqrt{15^2 - (4)(1)(-225)}}{2(1)} = \frac{15 \pm \sqrt{15^2 + (4)(15^2)}}{2(1)} = \frac{15 \pm \sqrt{(15^2)5}}{2} = \frac{15 \pm \sqrt{(15^2)}\sqrt{5}}{2} = \frac{15 \pm 15\sqrt{5}}{2}$$

Observe that $\frac{15 - 15\sqrt{5}}{2} < 0$ so the volume of the box can equal 13,500 when the dimension of the cross section is either 15 or $\frac{15 + 15\sqrt{5}}{2}$

A more general view of this problem is that we are finding the intersection of the graph of two functions; one whose rule is $c(x) = 13,500$ and the other whose rule is $V(x) = x^2(120 - 4x)$.



Graph of V with domain R and rule $V(x) = x^2(120 - 4x)$



Graph of V with domain (0, 30) and rule $V(x) = x^2(120 - 4x)$

Question

Analyze the function whose rule is $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

Analysis: $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$ is the rule for a fourth degree polynomial named f. Therefore the graph of f will be a smooth continuous curve with no sharp corners.

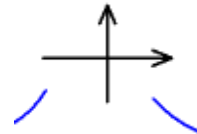
The graph will try to cross the x-axis four times.

The graph can have no more than four x-intercepts.

As $x \longrightarrow \infty$, $f(x) \longrightarrow -\infty$

As $x \longrightarrow -\infty$, $f(x) \longrightarrow -\infty$

The extremes of the graph look like



If $\frac{p}{q}$ is a zero of f, then

$$p \in \{\pm 1, \pm 2, \pm 4, \pm 8\} \text{ and } q \in \{\pm 1, \pm 2\}$$

$$\begin{aligned} \text{Then } \frac{p}{q} &\in \left\{ \pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{2}{1}, \pm \frac{2}{2}, \pm \frac{4}{1}, \pm \frac{4}{2}, \pm \frac{8}{1}, \pm \frac{8}{2} \right\} \\ &= \left\{ \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2} \right\} \end{aligned}$$

$$\text{Test1: } -2 + 13 - 21 + 2 + 8 = 0$$

So 1 is a zero of f and $(x - 1)$ is

a factor of $-2x^4 + 13x^3 - 21x^2 + 2x + 8$

Use ordinary long division of polynomials to determine the quotient.

$$\begin{array}{r} -2x^3 + 11x^2 - 10x - 8 \\ x-1 \overline{) -2x^4 + 13x^3 - 21x^2 + 2x + 8} \\ \underline{-2x^4 + 2x^3} \\ 11x^3 - 21x^2 + 2x + 8 \\ \underline{11x^3 - 11x^2} \\ -10x^2 + 2x + 8 \\ \underline{-10x^2 + 10x} \\ -8x + 8 \\ \underline{-8x - 8} \\ 0 \end{array}$$

$$\text{Therefore } f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$$

$$= (x-1)(-2x^3 + 11x^2 - 10x - 8)$$

Now we can continue searching for zeros of f by searching for zeros of the third degree function q whose rule is $q(x) = -2x^3 + 11x^2 - 10x - 8$.

The possible rational zeros of q are the same as the possible rational zeros of f.

$$\text{Test 1 (in q): } q(1) = -2 + 11 - 10 - 8 < 0$$

$$\text{Test -1 (in q): } q(-1) = -2(-1)^3 + 11(-1)^2 - 10(-1) - 8 = 2 + 11 + 10 - 8 > 0$$

According to the Intermediate Value Theorem, q must have a real zero between -1 and 1 . The zero guaranteed by the Intermediate Value Theorem might be irrational or it might be rational. If it is irrational we may not be able to find it. However, if it is rational it must be either $\frac{1}{2}$ or $-\frac{1}{2}$.

$$\begin{aligned} \text{Test } \frac{1}{2} \text{ (in } q\text{): } & -2\left(\frac{1}{2}\right)^3 + 11\left(\frac{1}{2}\right)^2 - 10\left(\frac{1}{2}\right) - 8 \\ & = -\frac{2}{8} + \frac{11}{4} - \frac{10}{2} - 8 = -\frac{1}{4} + \frac{11}{4} - \frac{20}{4} - \frac{32}{4} < 0 \end{aligned}$$

So $\frac{1}{2}$ is not a zero of q and therefore is not a zero of f .

$$\begin{aligned} \text{Test } -\frac{1}{2} \text{ (in } q\text{): } & -2\left(-\frac{1}{2}\right)^3 + 11\left(-\frac{1}{2}\right)^2 - 10\left(-\frac{1}{2}\right) - 8 \\ & = \frac{2}{8} + \frac{11}{4} + \frac{10}{2} - 8 = \frac{1}{4} + \frac{11}{4} + \frac{20}{4} - \frac{32}{4} = 0 \end{aligned}$$

So $-\frac{1}{2}$ is a zero of q and therefore is a zero of f .

Therefore $\left(x - \left(-\frac{1}{2}\right)\right) = \left(x + \frac{1}{2}\right)$ is a factor of the polynomial.

If $\left(x + \frac{1}{2}\right)$ is a factor, then $2\left(x + \frac{1}{2}\right) = (2x + 1)$ is a factor.

Use ordinary long division of polynomials to determine the quotient when $-2x^3 + 11x^2 - 10x - 8$ is divided by $2x + 1$

A Question About Polynomials

Question:

Suppose f is a third degree polynomial function.

Suppose it is known that:

1. f has a non-real complex zero
2. $f(1) < 0$
3. $f(2) > 0$

Is it possible for the graph of f to have an x -intercept in the ray $(2, \infty)$

Solution:

Because f is a third degree polynomial function it will have precisely three complex zeros. (Top of Page 293 and website Section 3.4)

Two of these zeros are not real numbers because complex zeros occur in conjugate pairs. (Page 297 and website Section 3.4).

The function f must have a zero in the interval $(1, 2)$ because $f(1) < 0$ and $f(2) > 0$. (Intermediate Value Theorem on Page 278 and website). This zero is a real number and in fact is the only real zero of the function f . If f were to have an x -intercept in the ray $(2, \infty)$ it would correspond to a real zero. This is impossible because we have determined that the only real zero is in the interval $(1, 2)$. It is impossible because $(1, 2) \cap (2, \infty) = \emptyset$.

Therefore it is impossible for f to have an x -intercept in the ray $(2, \infty)$.