Fractions -- Generalized

The intent in this short essay is to review in a unified manner all the important properties of fractions. The emphasis will be on those properties and operations which are the same for all types of fractions regardless of the kind of mathematical entity found in the numerator and denominator. The desire is to assist the reader to make the transition form 7th grade study of fractions to all those fractions which might be found in Algebra I through Calculus.

Important Background Facts

If you are unfamiliar with any of these background facts, it would be wise to review them in your textbook.

1. The set of integers is the set of counting numbers 1, 2, 3, • • • , their negatives • • • , - 3, - 2, - 1 and the number 0. The standard symbol for the set of integers is **Z**.

$$\mathbf{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$$

2. The set of rational numbers, denoted by Q, is the set of numbers which may be written as the ratio $\frac{a}{b}$ of integers where the denominator is of course not 0.

$$Q = \left\{ x \middle| x = \frac{p}{q},\ p \in Z,\ q \in Z,\ and\ q \neq 0 \right\}$$

3. The number 0 is the additive identity because adding 0 to any number produces a sum equal to the original number.
Symbolically: **a + 0 = a**

4. The number 1 is the multiplicative identity because multiplying any number by 1 produces a product equal to the original number.
Symbolically: **a·1 = a**

5. The opposite of a number is that number which is located on the opposite side of zero and equidistant from zero

5. The additive inverse of a real number a is a real number b such that the sum of a and b is the additive identity 0.
Symbolically: a and b are additive inverses of each other if a + b = 0
The sum of a number and its additive inverse is the additive identity.
The additive inverse of a real number is its opposite. **Symbolically:** a + (-a) = 0

6. The multiplicative inverse of a real number a is a real number b such that the product of a and b is the multiplicative identity.
Symbolically: a and b are multiplicative inverses of each other if ab = 1.
The product of a number and its multiplicative inverse is the multiplicative identity.
The multiplicative inverse of a real number is its reciprocal. **Symbolically:** $(a)\left(\frac{1}{a}\right) = 1$

7. All rational numbers are fractions but not all fractions are rational numbers.

Sets of Fractions

Definition: A fraction consists of a numerator (top), a denominator (bottom), and an indicated division of the numerator by the denominator.

Permitting the numerator and denominator to take on all possible values from a certain set forms different sets of fractions. The following sets of fractions are constructed in that manner.

If the numerator and denominator are permitted to be any integer (no 0 in the denominator), then we obtain the set of rational numbers. Some examples are listed below.

$$\frac{3}{4}, \frac{2}{5}, -\frac{3}{7}, \frac{8}{2}, \frac{3}{1}, \frac{4}{-9}, \frac{3}{17}, \frac{17}{3}, \frac{5}{12}, \frac{-2}{3}, -\frac{1}{2}$$

If the numerator and denominator are permitted to be any real number (no 0 in the denominator), then this set of fractions is much larger than the set of rational numbers. Every rational number is in this set, but many other fractions such as the ones shown below also appear in this set of fractions.

$$\frac{3}{\sqrt{2}}, \frac{\pi}{4}, \frac{\sqrt{5}}{\sqrt[5]{7}}, \frac{-\sqrt{5}}{\sqrt[3]{4}}, \frac{\pi}{2\sqrt{3}}, \frac{-3-\pi}{9}, \frac{3+\sqrt{5}}{1}, \frac{\pi\sqrt{7}+\sqrt{8}}{4}$$

If the numerator and denominator are permitted to be any algebraic expression (no 0 in the denominator), then this set of fractions is much larger than the previous set. Note that this set contains the previous set of fractions and therefore contains the set of rational numbers (the more familiar fractions). Some examples of fractions found in this set are listed below.

$$\frac{3}{4}, \frac{2}{5}, -\frac{3}{7}, \frac{8}{2}, \frac{3}{1}, \frac{4}{-9}, \frac{3}{17}, \frac{17}{3}, \frac{5}{12}, \frac{-2}{3}, -\frac{1}{2}$$

$$\frac{3}{\sqrt{2}}, \frac{\pi}{4}, \frac{\sqrt{5}}{\sqrt[5]{7}}, \frac{-\sqrt{5}}{\sqrt[3]{4}}, \frac{\pi}{2\sqrt{3}}, \frac{-3-\pi}{9}, \frac{3+\sqrt{5}}{1}, \frac{\pi\sqrt{7}+\sqrt{8}}{4}$$

$$\frac{a}{b}, \frac{3x-1}{k}, \frac{\sqrt{5}}{x^2+3}, \frac{3x^2+5x-2}{6x^3+4x-7}, \frac{\frac{3}{4}}{x^2}, \frac{\sqrt{3}x^2+7x-\pi}{ex^3-7+\sqrt{x^2+5}}$$

$$\frac{x^{\frac{2}{3}}}{y^{-\frac{5}{11}}}, \frac{\frac{1}{2}}{\frac{3}{4}}, \frac{\frac{-1}{2}}{\frac{-3}{-4}}, \frac{x^3y^{\frac{5}{2}}}{z}, \frac{\frac{x}{7}}{5}, \frac{\frac{x}{7}}{-5}, \frac{3x^2+5x-6}{x^2-6x+6}$$

Operations with and on fractions will be discussed in the context of this largest set of fractions and will therefore apply to all fractions. The mathematical operations/procedures remain the same for all kinds of fractions.

There are indeed other kinds of fractions, and the operations and procedures remain the same for those fractions as well. If the numerator and denominator are permitted to be any mathematical object for which division is defined, then we get a larger set of fractions. This set is considered too abstract to be the focus of discussion in this essay. However, even in this very large and very abstract set of fractions, the operations and procedures remain the same.

Operations with Fractions

There are only a few things we do with fractions; comparisons and operations. Operations are classified as either unary or binary operations. Unary operations are operations that involve only one fraction while binary operations are operations which involve two fractions.

The Binary Operations are:

- multiplication
- division
- addition
- subtraction

The Unary Operations are:

- Finding the additive inverse (opposite)
- Finding the multiplicative inverse (reciprocal)
- Expanding a fraction
- Reducing a fraction

Comparison of Fractions:

Those fractions which are Real Numbers may be represented as points on the Real Number Line. If two fractions are represented on the Real number line, then clearly the first is to the left of the second, equal to the second, or to the right of the second. The leftmost of two numbers on the Real Number Line is less than the other. This geometric view of real number fractions indicates that there should be an algebraic way to compare to Real Number fractions.

To compare two Real Number fractions with the same positive denominators, we need only compare the numerators.

Symbolically: If $\frac{a}{b}$ and $\frac{c}{b}$ are Real Number fractionswith positive denominators then

$\frac{a}{b} < \frac{c}{b}$ if and only if a < c

$\frac{a}{b} = \frac{c}{b}$ if and only if a = c

$\frac{a}{b} > \frac{c}{b}$ if and only if a > c

A few examples may clarify this method of comparison:

$\frac{3}{4} < \frac{9}{4}$ because 3 < 9.

$\frac{5}{8} > \frac{3}{8}$ because 5 > 3.

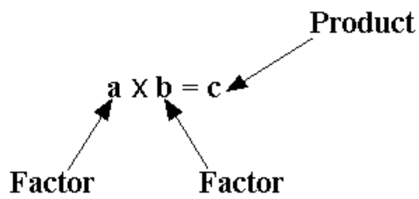
$\frac{15}{\sqrt{3}} = \frac{12+3}{\sqrt{3}}$ because 15 = 12 + 3.

To compare two Real Number fractions with different denominators, we expand the two fractions to fractions with the same positive denominators and then we need only compare the new numerators. This procedure will be revisited after a discussion of expanding fractions.

Multiplication of Fractions

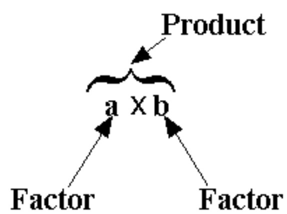
The simplest operation with fractions is the binary operation of multiplication. To understand the explanations and examples it is helpful to know the parts (and their names) of a multiplication problem.

Parts of a Multiplication Statement



Even if the multiplication is simply indicated but not performed, we refer to the product.

Parts of a Multiplication Expression



For example, in the expression (3)(12) = 36, 3 and 12 are factors and 36 is the product.
If we write (3)(12) then 3 and 12 are factors and we refer to (3)(12) as the product.
It is sometimes very convenient to refer to the product without actually computing it.
For example, it is convenient to be able to refer to the product of 739.87439 and 79376.235687 without the need to actually calculate it.

In the above schematics the symbols a and b can represent any kind of expression and in our present discussion they will represent fractions.

The process of multiplying two fractions is usually stated symbolically as

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \frac{ac}{bd}$$

First, note that in this "rule" the factors are $\frac{a}{b}$ and $\frac{\mathrm{c}}{\mathrm{d}}$ while the product is $\frac{\mathrm{ac}}{bd}$
Secondly, note that the product is a fraction – so the product of two fractions is itself a fraction.
Third, note that the "rule" states how the numerator and denominator of the product are computed.

Finally, in summary, here is what the "rule" tells us.

> The product of two fractions is a fraction.
> The numerator of the product is the product of the numerators.
> The denominator of the product is the product of the denominators.

Example 1: Multiply $\frac{3}{4}$ and $\frac{5}{7}$

Solution: The factors are $\frac{3}{4}$ and $\frac{5}{7}$

The numerator of the product is the product of 3 and 5 (the numerators) and is 15
The denominator of the product is the product of 4 and 7 (the denominators) and is 28
We would normally write this in the following fashion:

$$\left(\frac{3}{4}\right)\left(\frac{5}{7}\right) = \frac{(3)(5)}{(4)(7)} = \frac{15}{28}$$

Notice the indicated product (in the middle step) in the numerator and denominator.

RULE: It is always a good idea to put in the step that refers directly to the "rule" for multiplying fractions.

Always use the = symbol to indicate equality wherever it exists.

Example 2: Multiply $\frac{9}{5}$ and $\frac{2}{7}$

Solution: The factors are $\frac{9}{5}$ and $\frac{2}{7}$

The numerator of the product is the product of 9 and 2 (the numerators) and is 18
The denominator of the product is the product of 5 and 7 (the denominators) and is 35
We would normally write this in the following fashion:

$$\left(\frac{9}{5}\right)\left(\frac{2}{7}\right) = \frac{(9)(2)}{(5)(7)} = \frac{18}{35}$$

Notice the indicated product (in the middle step) in the numerator and denominator. It is always a good idea to put in the step that refers directly to the "rule" for multiplying fractions.

Always use the = symbol to indicate equality wherever it exists.

A word about the various symbols used to indicate multiplication is in order.
Two symbols ($\bullet$, $\times$) are frequently used to indicate multiplication, but the best way to indicate multiplication is to enclose the numbers inside grouping symbols (parenthesis) written next to each other (juxtaposition). Indicating multiplication with juxtaposition is illustrated in the above examples and everything that follows.

Example 3: Multiply $\frac{4}{12}$ and $\frac{9}{3}$

Solution: The factors are $\frac{4}{12}$ and $\frac{9}{3}$

The numerator of the product is the product of 4 and 9 (the numerators) and is 36
The denominator of the product is the product of 12 and 3 (the denominators) and is 36
We would normally write this in the following fashion:

$$\left(\frac{4}{12}\right)\left(\frac{9}{3}\right) = \frac{(4)(9)}{(12)(3)} = \frac{36}{36}$$

Notice the indicated product (in the middle step) in the numerator and denominator. It is always a good idea to put in the step that refers directly to the "rule" for multiplying fractions.

Always use the = symbol to indicate equality wherever it exists.

Example 4: $\left(\frac{5}{12}\right)\left(\frac{9}{2}\right)=\frac{(5)(9)}{(12)(2)}=\frac{45}{24}$

ALERT: There is no requirement to find and/or use common denominators when multiplying fractions.

The following examples show that multiplication of all types of fractions is performed in exactly the same manner as show in the previous examples:

RULE: **The numerator of the product is the product of the numerators**
The denominator of the product is the product of the denominators

Example 5: $\left(\frac{3x}{y}\right)\left(\frac{5}{x+2}\right)=\frac{(3x)(5)}{(y)(x+2)}=\frac{15x}{xy+2y}$

Example 6: $\left(\frac{\sqrt{2}}{x}\right)\left(\frac{y}{\sqrt{5}-x}\right)=\frac{\left(\sqrt{2}\right)(y)}{(x)\left(\sqrt{5}-x\right)}=\frac{\sqrt{2}y}{\sqrt{5}x-x^2}$

Example 7: $\left(\frac{3x^2-2}{x+1}\right)\left(\frac{5x-1}{\sqrt{x+4}}\right)=\frac{\left(3x^2-2\right)(5x-1)}{(x+1)\left(\sqrt{x+4}\right)}=\frac{15x^3-10x-3x^2+2}{x\sqrt{x+4}+\sqrt{x+4}}$

Example 8: $\left(\frac{x^{\frac{1}{2}}y^2}{2x}\right)\left(\frac{y^3}{x^5}\right)=\frac{\left(x^{\frac{1}{2}}y^2\right)\left(y^3\right)}{(2x)\left(x^5\right)}=\frac{x^{\frac{1}{2}}y^5}{2x^6}=\frac{y^5}{2x^{\frac{11}{2}}}$

Multiplicative Identity

It would seem natural, after completing the discussion of multiplication, to discuss one of the other binary operations (division, addition, subtraction). However, it is necessary to discuss one of the unary operations (finding the multiplicative inverse of a fraction) before moving on. In order to discuss multiplicative inverses, it is helpful to recall that the number 1 is the multiplicative identity and to recall some of its properties related to multiplication.

The number 1 is the **multiplicative identity** because multiplying any number by 1 produces a product equal to the original number.

Symbolically: **a·1 = a**

This fact about multiplication by 1 is true no matter what the other number is:

The product of 1 and a whole number is that whole number.
The product of 1 and an integer is that integer.
The product of 1 and a fraction is that fraction.
The product of 1 and an irrational number is that irrational number.
The product of 1 and a mixed number is that mixed number.
The product of 1 and a complex number is that complex number.

This fact about multiplication by 1 is true no matter how the number 1 is represented:

The product of $\frac{1}{1}$ and a number k is that number k.

The product of $\frac{5}{5}$ and a number k is that number k.

The product of $\frac{3\sqrt{2}}{3\sqrt{2}}$ and a number k is that number k.

The product of $(ax + 4)^0$ and a number k is that number k.

ALERT: There is only one multiplicative identity in the Real Number System

Multiplicative Inverses

ALERT: Every number other than 0 has a multiplicative inverse.

The **multiplicative inverse** of a real number a is a real number b such that the product of a and b is the multiplicative identity.

Symbolically: a and b are multiplicative inverses of each other if ab = 1

The product of a number and its multiplicative inverse is the multiplicative identity.
Every non-zero real number has a multiplicative inverse.
A number can have only one multiplicative inverse
If the number h is the multiplicative inverse of the number g, then g is the multiplicative inverse of h.
A number and its multiplicative inverse are multiplicative inverses of each other.
The multiplicative identity (the number 1) is the only real number which is its own multiplicative inverse.

The reciprocal of a real number is 1 divided by that number, so the reciprocal of a is $\frac{1}{a}$

The multiplicative inverse of a real number is its reciprocal. $(a)\left(\frac{1}{a}\right)=1$

The reciprocal of a fraction is the fraction formed by interchanging numerator and denominator.
The multiplicative inverse of a fraction is the fraction formed by interchanging numerator and denominator.

Here are a few examples of fractions of all types and their multiplicative inverses (reciprocals).

The multiplicative inverse of $\frac{2}{3}$ is $\frac{3}{2}$

The multiplicative inverse of 7 is $\frac{1}{7}$

The multiplicative inverse of $\frac{1}{6}$ is 6

The multiplicative inverse of $\frac{3}{\sqrt{5}}$ is $\frac{\sqrt{5}}{3}$

The multiplicative inverse of $\frac{x+6}{y-k}$ is $\frac{y-k}{x+6}$

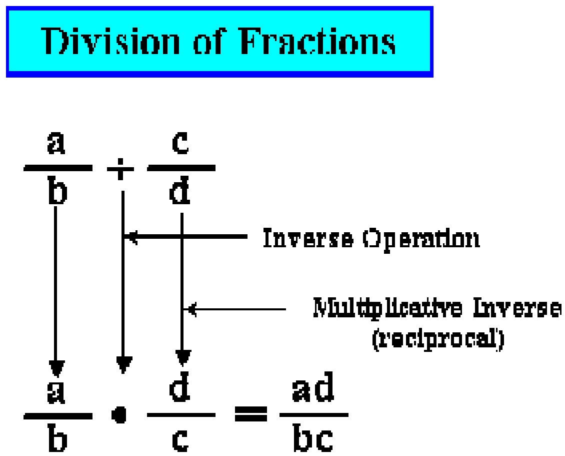
The multiplicative inverse of a^4 is a^{-4}

The multiplicative inverse of $\frac{3x^4-t}{y+t^{-3}}$ is $\frac{y+t^{-3}}{3x^4-t}$

The multiplicative inverse of $\frac{\sqrt{3}+2}{7-\sqrt{5}}$ is $\frac{7-\sqrt{5}}{\sqrt{3}+2}$

Division of Fractions

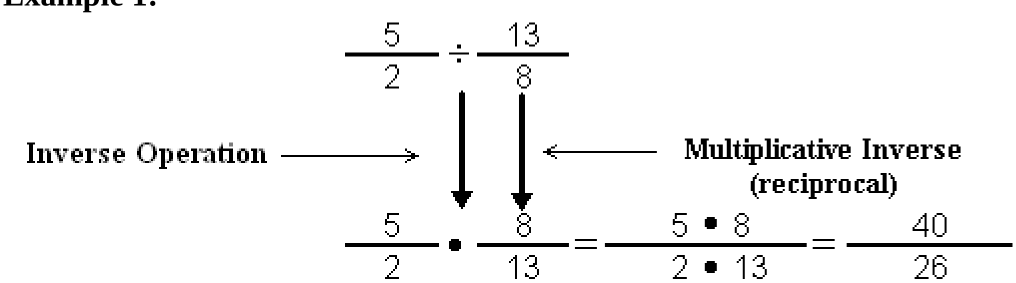
If $\frac{a}{b}$ and $\frac{c}{d}$ are any kind of fractions, the quotient of $\frac{a}{b}$ divided by $\frac{c}{d}$ is defined by converting the division to a multiplication as shown in the following diagram.



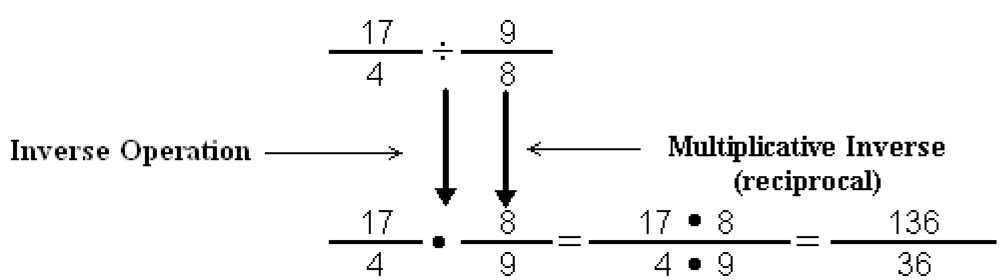
This diagram illustrates that to convert a division to a multiplication (the inverse operation), the dividend is unchanged, and the divisor is changed to its multiplicative inverse (reciprocal).

RULE: When faced with division of any kind of fractions, the division should be changed to a multiplication as described in the above diagram and then the multiplication should be carried out as explained in a previous section.

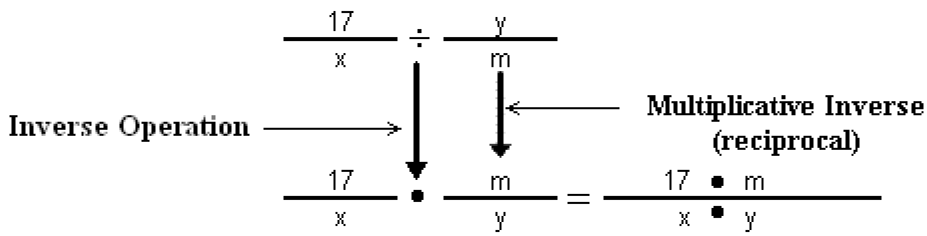
Example 1:



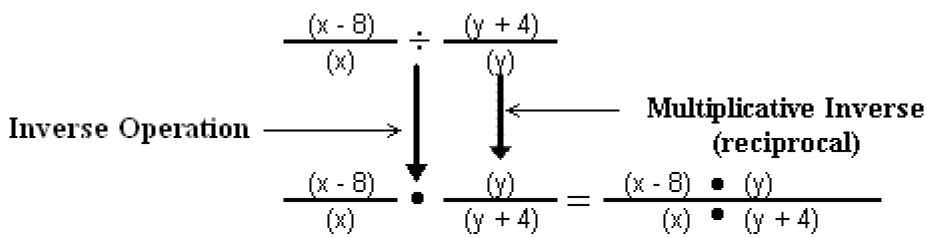
Example 2:



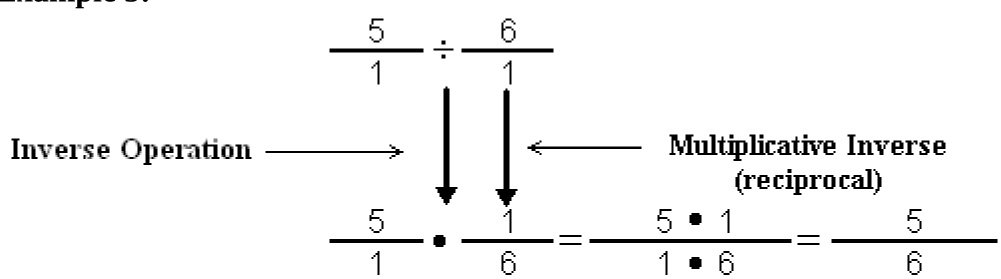
Example 3:



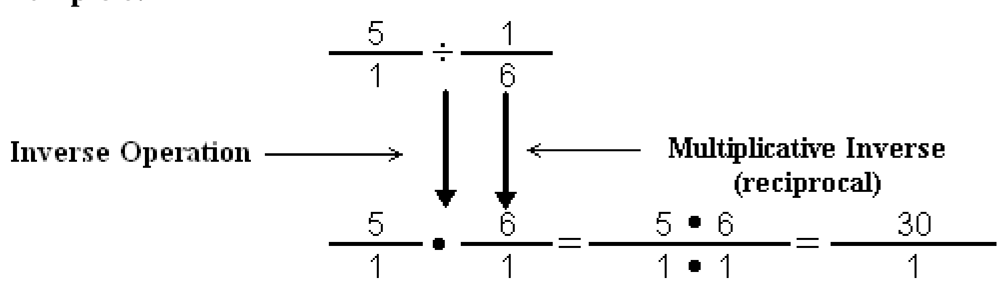
Example 4:



Example 5:



Example 6:



It is common practice to omit the diagram and simply write an equality which states that the indicated quotient is equal to the indicated product and then to complete the multiplication.

Example 7:

$$\frac{5}{3} \div \frac{8}{7} = \left(\frac{5}{3}\right)\left(\frac{7}{8}\right) = \frac{(5)(7)}{(3)(8)} = \frac{35}{24}$$

Example 8:

$$\frac{3x+1}{\sqrt{w}} \div \frac{5}{x-1} = \left(\frac{3x+1}{\sqrt{w}}\right)\left(\frac{x-1}{5}\right) = \frac{(3x+1)(x-1)}{(\sqrt{w})(5)} = \frac{3x^2-2x-1}{5\sqrt{w}}$$

Example 9:

$$\frac{x^{\frac{1}{3}}+5}{2x} \div \frac{7}{8} = \left(\frac{x^{\frac{1}{3}}+5}{2x}\right)\left(\frac{8}{7}\right) = \frac{8x^{\frac{1}{3}}+40}{14x}$$

Example 10:

$$11 \div \frac{2x+5y}{\sqrt[3]{x^2+y^2}} = 11\left(\frac{\sqrt[3]{x^2+y^2}}{2x+5y}\right) = \frac{11\sqrt[3]{x^2+y^2}}{2x+5y}$$

Addition of Fractions (Part I)

Learning to add fractions is a two step process: (1) Learn to add fractions with the same denominators; (2) Learn to convert other addition problems to problems like those in Step 1.

RULE: The sum of two fractions with the same denominator is a fraction with that common denominator and whose numerator is the sum of the numerators of the summands.

Symbolically: $\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$. Of course these fractions only make sense if b ≠ 0. Other than that there are no restrictions on a, b, or c , they can be just about any kind of mathematical expression.

Consider the following examples:

Example 1: $\frac{2}{9}+\frac{3}{9}=\frac{2+3}{9}=\frac{5}{9}$

Example 2: $\frac{3}{5}+\frac{\sqrt{7}}{5}=\frac{3+\sqrt{7}}{5}$

Example 3: $\frac{4}{\sqrt{3}}+\frac{x-1}{\sqrt{3}}=\frac{4+(x-1)}{\sqrt{3}}=\frac{x+3}{\sqrt{3}}$

Example 4: $\frac{3}{x^{\frac{4}{3}}}+\frac{7x}{x^{\frac{4}{3}}}=\frac{3+7x}{x^{\frac{4}{3}}}$

Example 5: $\frac{x-5\sqrt{w}}{3-\sqrt{2}}+\frac{y+5\sqrt{w}}{3-\sqrt{2}}=\frac{\left(x-5\sqrt{w}\right)+\left(y+5\sqrt{w}\right)}{3-\sqrt{2}}=\frac{x+y}{3-\sqrt{2}}$

Example 6: $\frac{1}{3\sqrt{5}+2x}+\frac{x}{3\sqrt{5}+2x}=\frac{x+1}{3\sqrt{5}+2x}$

Expanding Fractions

Before proceeding to a method for adding fractions with different denominators it is necessary to discuss one of the unary operations – expanding fractions.

Recall and be ready to use the fact that if any number is multiplied by 1 the product is that original number. On Page 1, it was pointed out that this property is why the number 1 is called the multiplicative identity. Additional discussion of the multiplicative identity is found on Pages 6 and 7. In particular, it is pointed out that the multiplicative identity can be written in many ways. Furthermore, regardless of its representation, multiplication with any other number produces a product equal to the other number. It might be helpful to look at the examples on Page 7.

It is frequently desirable to expand a fraction to another, but equal, fraction with a different denominator. For example, it might be desirable to write the fraction $\frac{7}{9}$ as a fraction with a denominator of 45. The only tool available for this expansion is multiplication by the multiplicative identity 1. It is important to remember that in this process we can represent 1 in any manner which serves our purpose.

RULE: To expand a fraction choose a fraction with equal numerator and denominator such that the product of its denominator and the denominator we are expanding is the desired new denominator. The product of this newly created fraction and the original fraction is the desired expanded fraction.

In the above example, choose a fraction $\frac{n}{n}$ such that 9n = 45, then multiply the original fraction by $\frac{n}{n}$.

The best way to organize this kind of problem is presented in the following examples:

Example: Expand $\frac{7}{9}$ to a fraction with a denominator of 45.

Solution: Step 1: Write what you wish to accomplish with space to fill in the missing step.

$$\frac{7}{9}=\left(\frac{7}{9}\right)\left(\frac{\ }{\ }\right)=\frac{\ }{45}$$

Step 2: Fill in the blanks in the above equality beginning with the denominator of the missing fraction.

$$\frac{7}{9}=\left(\frac{7}{9}\right)\left(\frac{5}{5}\right)=\frac{35}{45}$$

Example: Expand $\frac{7}{9}$ to a fraction with a denominator of $9\sqrt{5}$.

Solution: Step 1: Write what you wish to accomplish with space to fill in the missing step.

$$\frac{7}{9}=\left(\frac{7}{9}\right)\left(\frac{\ }{\ }\right)=\frac{\ }{9\sqrt{5}}$$

Step 2: Fill in the blanks in the above equality beginning with the denominator of the missing fraction. Clearly we must multiply 9 and $\sqrt{5}$ to obtain $9\sqrt{5}$.

$$\frac{7}{9}=\left(\frac{7}{9}\right)\left(\frac{\sqrt{5}}{\sqrt{5}}\right)=\frac{7\sqrt{5}}{9\sqrt{5}}$$

Example: Expand $\frac{x+3}{\sqrt{7}}$ to a fraction with a denominator of 7.

Solution: Step 1: Write what you wish to accomplish with space to fill in the missing step.

$$\frac{x+3}{\sqrt{7}}=\left(\frac{x+3}{\sqrt{7}}\right)\left(\frac{\ }{\ }\right)=\frac{\ }{7}$$

Step 2: Fill in the blanks in the above equality beginning with the denominator of the missing fraction. Clearly we must multiply $\sqrt{7}$ and $\sqrt{7}$ to obtain 7.

$$\frac{x+3}{\sqrt{7}} = \left(\frac{x+3}{\sqrt{7}}\right)\left(\frac{\sqrt{7}}{\sqrt{7}}\right) = \frac{\sqrt{7}\left(x+3\right)}{7}$$

Example: Expand $\frac{x-5}{2x+1}$ to a fraction with a denominator of (x + 5)(2x + 1).

Solution: **Step 1:** Write what you wish to accomplish with space to fill in the missing step.

$$\frac{x-5}{2x+1} = \left(\frac{x-5}{2x+1}\right)\left(\frac{\ }{\ }\right) = \frac{\ }{(2x+1)(x+5)}$$

Step 2: Fill in the blanks in the above equality beginning with the denominator of the missing fraction. Clearly we must multiply (x + 5) and (2x + 1).

$$\frac{x-5}{2x+1} = \left(\frac{x-5}{2x+1}\right)\left(\frac{x+5}{x+5}\right) = \frac{x^2-25}{(2x+1)(x+5)}$$

Example: Expand $\frac{3x+2}{xy}$ to a fraction with a denominator of x^2y^3.

Solution: **Step 1:** Write what you wish to accomplish with space to fill in the missing step.

$$\frac{3x+2}{xy} = \left(\frac{3x+2}{xy}\right)\left(\frac{\ }{\ }\right) = \frac{\ }{x^2y^3}$$

Step 2: Fill in the blanks in the above equality beginning with the denominator of the missing fraction.

$$\frac{3x+2}{xy} = \left(\frac{3x+2}{xy}\right)\left(\frac{xy^2}{xy^2}\right) = \frac{xy^2(3x+2)}{x^2y^3}$$

Example: Expand $\frac{5}{12}$ to a fraction with a denominator of 8

Solution: **Step 1:** Write what you wish to accomplish with space to fill in the missing step.

$$\frac{5}{12} = \left(\frac{5}{12}\right)\left(\frac{\ }{\ }\right) = \frac{\ }{8}$$

Step 2: Fill in the blanks in the above equality beginning with the denominator of the missing fraction.

$$\frac{5}{12} = \left(\frac{5}{12}\right)\left(\frac{\frac{2}{3}}{\frac{2}{3}}\right) = \frac{\left(\frac{10}{3}\right)}{8}$$

Addition of Fractions (Part II)

RULE: To add two fractions with different denominators select a convenient denominator, expand both fractions to a fraction with the selected denominator and then add the two fractions with the same denominator as described in an earlier section.

Example 1: $\frac{3}{4}+\frac{2}{5}$

Solution: Select 20 as a new denominator and expand both fractions to a fraction with 20 as its denominator.

$$\frac{3}{4}=\left(\frac{3}{4}\right)\left(\frac{5}{5}\right)=\frac{15}{20} \qquad \text{and} \qquad \frac{2}{5}=\left(\frac{2}{5}\right)\left(\frac{4}{4}\right)=\frac{8}{20}$$

Then we have $\frac{3}{4}+\frac{2}{5}=\frac{15}{20}+\frac{8}{20}=\frac{15+8}{20}=\frac{23}{20}$

Example 2: $\frac{7}{11}+\frac{3}{\sqrt{5}}$

Solution: Select $11\sqrt{5}$ as a new denominator and expand both fractions to a fraction with $11\sqrt{5}$ as its denominator

Then we have $\frac{7}{11}+\frac{3}{\sqrt{5}}=\left(\frac{7}{11}\right)\left(\frac{\sqrt{5}}{\sqrt{5}}\right)+\left(\frac{3}{\sqrt{5}}\right)\left(\frac{11}{11}\right)=\frac{7\sqrt{5}+33}{11\sqrt{5}}$

Example 3: $\frac{5}{6}+\frac{7}{15}$

Solution: Select 30 as a new denominator and expand both fractions to a fraction with 30 as its denominator.

Then we have $\frac{5}{6}+\frac{7}{15}=\left(\frac{5}{6}\right)\left(\frac{5}{5}\right)+\left(\frac{7}{15}\right)\left(\frac{2}{2}\right)=\frac{25}{30}+\frac{14}{30}=\frac{25+14}{30}=\frac{39}{30}$

Example 4: $\frac{4}{6x^2\sqrt{11}}+\frac{3+2\sqrt{5}}{9x}$

Solution: Select $18x^2\sqrt{11}$ as a new denominator and expand both fractions to a fraction with $18x^2\sqrt{11}$ as its denominator.

Then we have

$$\frac{4}{6x^2\sqrt{11}}+\frac{3+2\sqrt{5}}{9x}=\left(\frac{4}{6x^2\sqrt{11}}\right)\left(\frac{3}{3}\right)+\left(\frac{3+2\sqrt{5}}{9x}\right)\left(\frac{2x\sqrt{11}}{2x\sqrt{11}}\right)=\frac{12}{18x^2\sqrt{11}}+\frac{\left(2x\sqrt{11}\right)\left(3+2\sqrt{5}\right)}{6x^2\sqrt{11}}=\frac{12+6\sqrt{11}x+4\sqrt{55}x}{6x^2\sqrt{11}}$$

Example 5: $\frac{x}{x+2}+\frac{3x}{(x+2)(x-1)}$

Solution: Select (x + 2)(x – 1) as a new denominator and expand both fractions to a fraction with (x + 2)(x – 1) as its denominator.

Then we have

$$\frac{x}{x+2}+\frac{3x}{(x+2)(x-1)}=\left(\frac{x}{x+2}\right)\left(\frac{x-1}{x-1}\right)+\left(\frac{3x}{(x+2)(x-1)}\right)=\frac{x(x-1)}{(x+2)(x-1)}+\frac{3x}{(x+2)(x-1)}=\frac{x^2+2x}{(x+2)(x-1)}$$

Three Signs of a Fraction

Look carefully at any real number fraction $\frac{numerator}{deno\min ator}$ and you can see three numbers; the fraction itself, the numerator, and the denominator. For each of these three numbers it is completely reasonable to speak of their opposites. So we may speak of **the opposite of the fraction itself, the opposite of the numerator, or the opposite of the denominator**. Recall that the opposite of a number is represented by placing a – symbol in front of the number. This gives rise to three locations for a – symbol to appear in a fraction:

$$-\frac{-a}{-b}$$

The appearance or non-appearance of the – symbol gives the following two strings of equalities:

$$\frac{a}{b} = \frac{-a}{-b} = -\frac{-a}{b} = -\frac{a}{-b}$$

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b} = -\frac{-a}{-b}$$

The first string of equalities shows four ways to write a fraction $\frac{a}{b}$

The second string of equalities shows four ways to write the opposite of a fraction $\frac{a}{b}$

These different representations are also valid for all other kinds of fractions.

Example 1: Four ways to write the fraction $\frac{3}{4}$ are: $\frac{3}{4} = \frac{-3}{-4} = -\frac{-3}{4} = -\frac{3}{-4}$

Four ways to write the opposite of $\frac{3}{4}$ are: $-\frac{3}{4} = \frac{-3}{4} = \frac{3}{-4} = -\frac{-3}{-4}$

Example 2: Four ways to represent the fraction $\frac{-2}{5}$ are: $\frac{-2}{5} = \frac{-(-2)}{-5} = -\frac{-(-2)}{5} = -\frac{-2}{-5}$

Removing parenthesis yields: $\frac{-2}{5} = \frac{2}{-5} = -\frac{2}{5} = -\frac{-2}{-5}$

Four ways to represent the opposite of $\frac{-2}{5}$ are: $-\frac{-2}{5} = \frac{-(-2)}{5} = \frac{-2}{-5} = -\frac{-(-2)}{-5}$

Removing parenthesis yields: $-\frac{-2}{5} = \frac{2}{5} = \frac{-2}{-5} = -\frac{2}{-5}$

Example 3: Four ways to represent the fraction $\frac{\sqrt{5}}{x-1}$ are: $\frac{\sqrt{5}}{x-1} = \frac{-\sqrt{5}}{-(x-1)} = -\frac{-\sqrt{5}}{x-1} = -\frac{\sqrt{5}}{-(x-1)}$

Removing parenthesis yields: $\frac{\sqrt{5}}{x-1} = \frac{-\sqrt{5}}{1-x} = -\frac{-\sqrt{5}}{x-1} = -\frac{\sqrt{5}}{1-x}$

Four ways to represent the opposite of $\frac{\sqrt{5}}{x-1}$ are: $-\frac{\sqrt{5}}{x-1} = \frac{-\sqrt{5}}{x-1} = \frac{\sqrt{5}}{-(x-1)} = -\frac{-\sqrt{5}}{-(x-1)}$

Removing parenthesis yields: $-\frac{\sqrt{5}}{x-1} = \frac{-\sqrt{5}}{x-1} = \frac{\sqrt{5}}{1-x} = -\frac{-\sqrt{5}}{1-x}$

Example 4: Four ways to represent the fraction $\frac{2x-1}{x-5}$ are:

$$\frac{2x-1}{x-5} = \frac{-(2x-1)}{-(x-5)} = -\frac{-(2x-1)}{x-5} = -\frac{2x-1}{-(x-5)}$$

Removing parenthesis yields: $\frac{2x-1}{x-5} = \frac{1-2x}{5-x} = -\frac{1-2x}{x-5} = -\frac{2x-1}{5-x}$

Four ways to represent the opposite of $\frac{2x-1}{x-5}$ are:

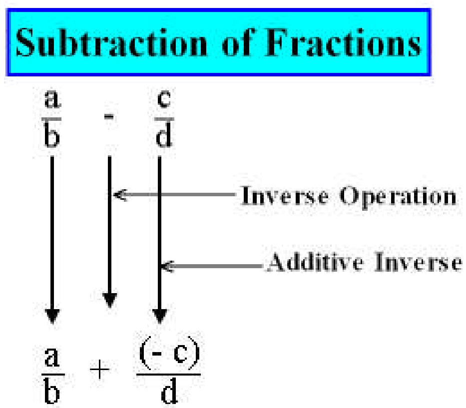
$$-\frac{2x-1}{x-5} = \frac{-(2x-1)}{x-5} = \frac{2x-1}{-(x-5)} = -\frac{-(2x-1)}{-(x-5)}$$

Removing parenthesis yields: $-\frac{2x-1}{x-5}=\frac{1-2x}{x-5}=\frac{2x-1}{5-x}=-\frac{1-2x}{5-x}$

The previous examples illustrate some of the many and varied forms we can use for a fraction and its opposite. In the next section we are particularly concerned with writing the opposite of a fraction.

Subtraction of Fractions

If $\frac{a}{b}$ and $\frac{c}{d}$ are any kind of fractions, the difference of $\frac{c}{d}$ subtracted from $\frac{a}{b}$ is defined by converting the subtraction to an addition as shown in the following diagram.



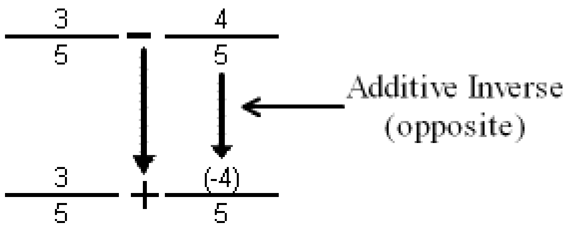
This diagram illustrates that to convert a subtraction to an addition (the inverse operation), the minuend is unchanged, and the subtrahend is changed to its additive inverse (opposite).

RULE: When faced with subtraction of any kind of fractions the subtraction should be changed to an addition as described in the above diagram and then the addition should be carried out as explained in a previous section.

A few examples will clarify the process. **Careful selection of the proper representation for the opposite of a fraction can expedite subtraction of fractions**. That technique is illustrated in some of the following examples.

Example 1: Subtract: $\frac{3}{5}-\frac{4}{5}$

Solution: Begin by changing the problem to addition



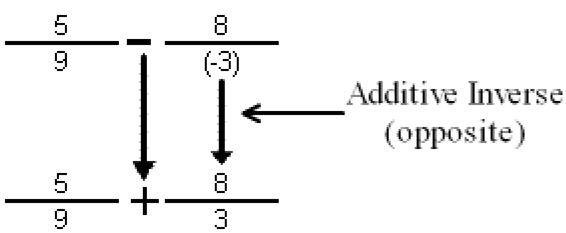
The do the addition $\frac{3}{5}+\frac{(-4)}{5}=\frac{3+(-4)}{5}=\frac{-1}{5}$

Therefore $\frac{3}{5}-\frac{4}{5}=\frac{-1}{5}$

Example 2: Subtract: $\frac{5}{9}-\frac{8}{(-3)}$

Solution: Begin by changing the problem to addition.

Observe that one way to write the opposite of $\frac{8}{(-3)}$ *is* $\frac{8}{-(-3)}=\frac{8}{3}$



Now do the addition: $\frac{5}{9}+\frac{8}{3}=\frac{5}{9}+\frac{24}{9}=\frac{29}{9}$

Therefore $\frac{5}{9}-\frac{8}{(-3)}=\frac{29}{9}$

Example 3: Subtract: $\frac{5}{x}-\frac{y}{(-x)}$

Solution: Begin by changing the problem to addition.

$$\frac{5}{x} \quad - \quad \frac{y}{(-x)}$$

$$\downarrow \quad \downarrow$$

$$\frac{5}{x} \quad + \quad \frac{y}{x}=\frac{y+5}{x}$$

Example 4: Subtract: $\frac{2x-1}{3x-5}-\frac{5-7x}{5-3x}$

Solution: Begin by changing the problem to addition.

$$\frac{2x-1}{3x-5}-\frac{5-7x}{5-3x}$$

$$\downarrow \quad \downarrow$$

$$\frac{2x-1}{3x-5} \quad + \quad \frac{5-7x}{3x-5}=\frac{(2x-1)+(5-7x)}{3x-5}=\frac{-5x+4}{3x-5}$$

Example 5: Subtract: $\frac{3x+y}{2x-5}-\frac{\sqrt{2}x-5}{y+3}$

Solution: Begin by changing the problem to addition

$$\frac{3x+y}{2x-5}-\frac{\sqrt{2}x-5}{y+3}$$

$$\downarrow \quad \downarrow$$

$$\frac{3x+y}{2x-5}+\frac{5-\sqrt{2}x}{y+3}=\left(\frac{3x+y}{2x-5}\right)\left(\frac{y+3}{y+3}\right)+\left(\frac{5-\sqrt{2}x}{y+3}\right)\left(\frac{2x-5}{2x-5}\right)=\frac{(3x+y)(y+3)+\left(5-\sqrt{2}x\right)(2x-5)}{(y+3)(2x-5)}$$

$$\frac{3xy+9x+y^2+3y+10x-25-2\sqrt{2}x^2+5\sqrt{2}x}{(y+3)(2x-5)}=\frac{y^2-2\sqrt{2}x^2+(19+5\sqrt{2})x+3y+3xy-25}{(y+3)(2x-5)}$$

Reduction of Fractions

Reducing a fraction refers to a process which identifies and removes factors which are common to both the numerator and denominator.

Reducing is one of the unary operations listed earlier and is the reverse of expanding a fraction. Just as expanding a fraction depends on the multiplicative properties of the number 1, so also reducing a fraction depends on those very same multiplicative properties of the number 1. They are repeated for review here.

Recall and be ready to use the fact that if any number is multiplied by 1 the product is that original number. On Page 1, it was pointed out that this property is why the number 1 is called the multiplicative identity. Additional discussion of the multiplicative identity is found on Pages 6 and 7. In particular, it is pointed out that the multiplicative identity can be written in many ways. Furthermore, regardless of its representation, multiplication with any other number produces a product equal to the other number. It might be helpful to look at the examples on Page 7.

It is frequently desirable to reduce a fraction to another, but equal, fraction with a different denominator. For example, it might be desirable to write the fraction $\frac{105}{45}$ as a fraction with a denominator of 15. This would yield the fraction $\frac{35}{15}$. In other situations it is desirable to reduce a fraction so that there are no factors common to the numerator and denominator. When this is done, we say that the fraction has been reduced to lowest terms. The fraction $\frac{105}{45}$ reduced to lowest terms is $\frac{7}{3}$.

RULE: The procedure for reducing a fraction is to identify the factors of the numerator and denominator, group them to form representations of the number 1, write the fraction as a product which isolates the representations of the number 1, and then delete the factors of 1 from the product.

The following examples will illustrate.

Example 1: Completely reduce the fraction $\frac{105}{45}$

Solution: $\frac{105}{45} = \frac{(3)(5)(7)}{(3)(3)(5)} = \left(\frac{3}{3}\right)\left(\frac{5}{5}\right)\left(\frac{7}{3}\right) = (1)(1)\left(\frac{7}{3}\right) = \frac{7}{3}$

Example 2: Completely reduce the fraction $\frac{x^2-2x}{(x+2)(x-2)}$

Solution: $\frac{x^2-2x}{(x+2)(x-2)} = \frac{x(x-2)}{(x+2)(x-2)} = \left(\frac{x}{x+2}\right)\left(\frac{x-2}{x-2}\right) = \frac{x}{x+2}$

Example 3: Completely reduce the fraction $\frac{x^2+6x-40}{10+x}$

Solution: $\frac{x^2+6x-40}{10+x} = \frac{(x+10)(x-4)}{x+10} = \left(\frac{x+10}{x+10}\right)(x-4) = x-4$

Example 4: Completely reduce the fraction $\frac{18x^3y^2}{3xz^4}$

Solution: $\frac{18x^3y^2}{3xz^4} = \left(\frac{18}{3}\right)\left(\frac{x^3}{x}\right)\left(\frac{y^2}{z^4}\right) = (6)(x^2)\left(\frac{y^2}{z^4}\right) = \frac{6x^2y^2}{z^4}$

Notice that in this example the rules for exponents facilitated the reduction.

Example 5: Completely reduce the fraction $\frac{3x^2-5x-2}{6x^3+2x^2+3x+1}$

Solution:

$$\frac{3x^2-5x-2}{6x^3+2x^2+3x+1} = \frac{(3x+1)(x-2)}{2x^2(3x+1)+(3x+1)} = \frac{(3x+1)(x-2)}{(3x+1)(2x^2+1)} = \left(\frac{3x+1}{3x+1}\right)\left(\frac{x-2}{2x^2+1}\right) = \frac{x-2}{2x^2+1}$$

Example 6: Perform the multiplication and completely reduce the product

$$\left(\frac{2x^3+16}{6x^2+12x}\right)\left(\frac{5}{x^2-2x+4}\right)$$

Solution:

$$\left(\frac{2x^3+16}{6x^2+12x}\right)\left(\frac{5}{x^2-2x+4}\right) = \left(\frac{2(x+2)(x^2-2x+4)}{6x(x+2)}\right)\left(\frac{5}{x^2-2x+4}\right)$$

$$= \frac{(2)(5)(x+2)(x^2-2x+4)}{(2)(3)(x)(x+2)(x^2-2x+4)} = \left(\frac{2}{2}\right)\left(\frac{x+2}{x+2}\right)\left(\frac{(x^2-2x+4)}{(x^2-2x+4)}\right)\left(\frac{(5)}{(3)(x)}\right) = \frac{5}{3x}$$

Observe that it is much easier to do the reduction (cancellation) before multiplying