Law of Trichotomy and Boundary Equations

Law of Trichotomy: For any two real numbers a and b, exactly one of the following is true.

i. $a < b$
ii. $a = b$
iii. $a > b$

The Law of Trichotomy is a formal statement of a property which most of us would consider to be quite obvious; when comparing two numbers; they are equal, the first is less than the second, or the first is greater than the second. The purpose of the formal statement here is to call attention to the obvious fact and to make it available for use with algebraic quantities which represent real numbers.

Example 1: During a consideration of the two linear algebraic expressions 3x + 5 and -2x + 7, the Law of Trichotomy reminds us that there exists three distinct possibilities;

i. $3x + 5 < -2x + 7$
ii. $3x + 5 = -2x + 7$
iii. $3x + 5 > -2x + 7$

Consequently we learn to solve the equation and both inequalities. Recall some of the observations made during the many practice exercises for solving linear equations and inequalities.

i. The graph of a linear equation in one variable is a point on the real number line.
ii. The graph of a linear inequality in one variable is a ray on the real number line.

What may not have been observed is that the ray which is the graph of a linear inequality in one variable begins at the graph of the equation and extends infinitely far toward the right or the left and that the graph of the other inequality begins at the same point and extends infinitely far in the other direction.

Another, possibly more understandable, way to state this is: The graph of a linear equation in one variable divides the real number line into two rays, one of which is the graph of one of the corresponding inequalities and the other is the graph of the other inequality.

Refer to Example 1. The solution set for the equation is $\left\{\frac{2}{5}\right\}$.

Fig.1

The graph of the equation $3x + 5 = -2x + 7$ is shown in Fig. 1.

Fig. 2

Clearly this graph of the equation divides the real number line into a point and two rays as shown in Fig. 2.The blue ray is the graph of one of the inequalities and the red ray is the graph of the other inequality. In this example the blue ray is the graph of $3x + 5 < -2x + 7$ and the red ray is the graph of $3x + 5 > -2x + 7$.

We can determine if the blue ray is the solution set to one of the inequalities by testing just one number from the ray in the inequality. Refer to the above example. To determine if the blue ray is the solution to the inequality $3x + 5 > -2x + 7$ we need only test one number from the blue ray in $3x + 5 > -2x + 7$. The number 0 is in the blue ray and is easy to test. Substituting 0 into $3x + 5 > -2x + 7$ yields $5 > 7$ which is false. This allows a number of conclusions;

i. The blue ray is not the solution set for $3x + 5 > -2x + 7$.
ii. The red ray (the other one) is the solution set for $3x + 5 > -2x + 7$
iii. The blue ray is the solution set for $3x + 5 < -2x + 7$.

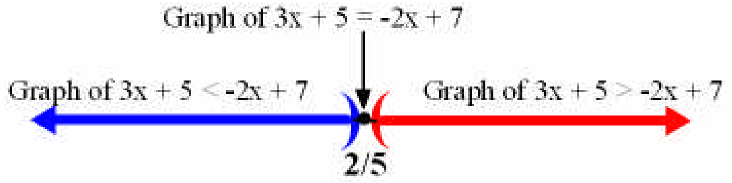

Fig. 3

The following observations can be generalized to many other situations. In particular they will apply to equations and inequalities involving nothing but polynomials.

i. When considering a conditional equation or inequality, the Law of Trichotomy dictates that we also consider the other two corresponding equations and/or inequalities.
ii. The graph of the equation is a boundary between the graphs of the corresponding inequalities. For that reason, the equation is sometimes called the boundary equation for the inequalities.
iii. Testing any single number from one of the rays in either inequality determines whether that ray is the solution set for that inequality.

Extending to Quadratic Inequalities in One Variable

Example 2: Suppose it is required to solve the inequality $x^2 + x - 6 < 0$.
Discussion and Solution:
When considering the inequality $x^2 + x - 6 < 0$ the Law of Trichotomy dictates that we be aware of the equality $x^2 + x - 6 = 0$ as well as the inequality $x^2 + x - 6 > 0$. As in the previous discussion, the equation $x^2 + x - 6 = 0$ is sometimes called the boundary equation because its graph is the boundary between the graphs of the two inequalities. Our strategy will be to graph the equation and test numbers from the various resulting rays and intervals formed by that graph.
Factoring and The Zero Factor Property show the solution set of the equation $x^2 + x - 6 = 0$ to be {2, -3}. We can now sketch the graph of the equation $x^2 + x - 6 = 0$ as shown in Fig. 4.

Fig. 4

The graph of the equation $x^2 + x - 6 = 0$ divides the real line into an interval (-3, 2) (green) and two rays (blue $(-\infty, -3)$ and red $(2, \infty)$).

- The interval is part of the solution set for either $x^2 + x - 6 < 0$ or $x^2 + x - 6 > 0$.
- The blue ray $(-\infty, -3)$ is part of the solution set for $x^2 + x - 6 < 0$ or $x^2 + x - 6 > 0$.
- The red ray $(2, \infty)$ is part of the solution set for $x^2 + x - 6 < 0$ or $x^2 + x - 6 > 0$.

We need to test one number from each of the rays and the interval in either of the inequalities.

Test 0 from the interval (-3, 2) in the inequality $x^2 + x - 6 < 0$. When 0 is substituted into $x^2 + x - 6 < 0$ we obtain $-6 < 0$; a true statement. Therefore the interval is part of the solution set for $x^2 + x - 6 < 0$.
Test – 4 from the ray $(-\infty, -3)$ in the inequality $x^2 + x - 6 < 0$. When – 4 is substituted into $x^2 + x - 6 < 0$ we obtain $16 - 4 - 6 < 0$; a false statement. Therefore – 4, and consequently no number in the ray $(-\infty, -3)$, is a solution of the inequality $x^2 + x - 6 < 0$.
It now follows from the Law of Trichotomy that every number in the ray $(-\infty, -3)$ is a solution of the other inequality $x^2 + x - 6 > 0$. Therefore the ray $(-\infty, -3)$ is part of the solution set for $x^2 + x - 6 > 0$.
Test 3 from the ray $(2, \infty)$ in the inequality $x^2 + x - 6 > 0$. (note I switched inequalities). When 3 is substituted into $x^2 + x - 6 > 0$ we obtain $9 + 3 - 6 >$ a true statement. Consequently the ray $(2, \infty)$ is part of the solution set for $x^2 + x - 6 < 0$.

A summary of the test results shows that the solution set for the inequality $x^2 + x - 6 > 0$ is $(-\infty, -3) \cup (2, \infty)$ and the solution set for $x^2 + x - 6 > 0$ is the interval (-3, 2). Illustrated in Fig. 5.

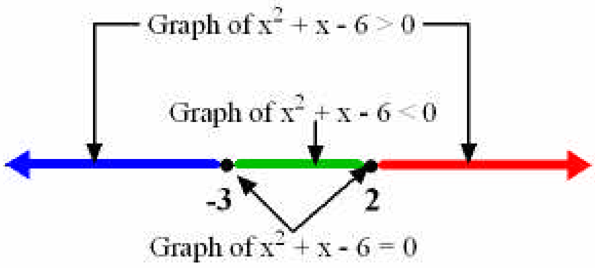

Fig. 5

Rule for Quadratics: When working with quadratic equations and inequalities in one variable;

- If the equation has two real solutions, the resulting interval will be the solution set for one of the inequalities and the union of the rays will be the solution set for the other inequality.
- If the equation has one real solution the union of the resulting two rays will be the solution set for one of the inequalities and the solution set for the other inequality is the empty set $\varnothing$.
- If the equation has no real solution, the solution set for one of the inequalities is the set of real numbers **R** and the solution set for the other inequality is the empty set $\varnothing$.

Rule: The above process may be used to solve any polynomial inequality in one variable.

This will be illustrated in Example 3.
Example 3: Solve the inequality $x(x - 2)(x + 3)(x + 3)(x + 4) < 0$.

Discussion and Solution:

Generally fifth degree polynomial inequalities in one variable will not be solvable, but when we can factor it, as in this case, we can solve the inequality

When considering the inequality $x(x-2)(x+3)(x+3)(x+4) < 0$ the Law of Trichotomy dictates that we be aware of the equality $x(x-2)(x+3)(x+3)(x+4) = 0$ as well as the inequality $x(x-2)(x+3)(x+3)(x+4) > 0$.The equation $x(x-2)(x+3)(x+3)(x+4) = 0$ is called the boundary equation because its graph is the boundary between the graphs of the two inequalities.

Our strategy will be to graph the equation and test numbers from the various resulting rays and intervals formed by that graph.

The Zero Factor Property shows the solution set of the equation $x(x-2)(x+3)(x+3)(x+4) = 0$ to be $\{-4, -3, 0, 2\}$. A sketch of the graph of the equation $x(x-2)(x+3)(x+3)(x+4) = 0$ as shown in Fig. 6. Observe the rays and intervals are;

$(-\infty, -4)$, $(-4, -3)$, $(-3, 0)$, $(0, 2)$, and $(2, \infty)$

Fig. 6

The solution set for the two inequalities can be determined by testing a number from each of the rays and intervals. However there is a simpler approach which depends on the fact that it is simple to determine the sign of a linear expression in an interval.

A convenient and effective way to organize the test is with a table as illustrated in Fig. 7.

	(-∞, -4)	(-4, -3)	(-3, 0)	(0, 2)	(2, ∞)
sign of x	—	—	—	+	+
sign of x - 2	—	—	—	—	+
sign of (x + 3)(x+3)	+	+	+	+	+
sign of x + 4	—	+	+	+	+
sign of x(x-2)(x+3)(x+3)(x+4)	—	+	+	—	+

Fig. 7

The final row of this table shows where the polynomial is positive and where it is negative. We therefore conclude that:

The solution set for $x(x-2)(x+3)(x+3)(x+4) = 0$ is $\{-4, -3, 0, 2\}$.

The solution set for $x(x-2)(x+3)(x+3)(x+4) > 0$ is $(-4, -3) \cup (-3, 0) \cup (2, \infty)$.

The solution set for $x(x-2)(x+3)(x+3)(x+4) < 0$ is $(-\infty, -4) \cup (0, 2)$.

Important Notes About Polynomial Inequalities: As a first step when working with polynomial inequalities one should write the inequality in the form

$$a_n x^n + a_{n-1} x^{n-1} + \bullet \bullet \bullet + a_1 x + a_0 < 0 \quad \text{or}$$
$$a_n x^n + a_{n-1} x^{n-1} + \bullet \bullet \bullet + a_1 x + a_0 > 0$$

Writing a polynomial inequality in the desired form may be done using the three familiar, fundamental, elementary properties of inequalities.

(1): If any expression is added to both sides of an inequality the resulting inequality is equivalent to the original inequality.

(2): If both sides of an inequality are multiplied by the same positive real number, the resulting inequality is equivalent to the original inequality.

(3): If both sides of an inequality are multiplied by the same negative real number and the inequality symbol is reversed, the resulting inequality is equivalent to the original inequality.

As a first step when working with polynomial inequalities one should find the real solutions of the corresponding boundary equation and factor the polynomial into a product of linear and quadratic factors. Although this is theoretically always possible, it is generally very difficult and frequently impossible to do. When the real solutions of the boundary equation cannot be found, other methods, beyond elementary algebra, must be employed.

A table such as shown in Fig. 7 is especially useful and almost essential when the expression involved is complex.