Analysis of Rational Functions

Question:
Analyze and graph the rational function whose rule is \( f(x) = \frac{2x - 3}{x - 4} \).

Analysis:

a) Determine the domain of \( f \)

When the domain of a function is not explicitly stated, convention dictates that the domain is the largest subset of the real numbers for which the rule makes sense (is defined).

In the case of a rational function the rule for \( f \) is defined for all real numbers except for zeros of the denominator.

In this case the rule for \( f \) is defined for all real numbers except when \( x - 4 = 0 \).

The domain of \( f \) is all real numbers except 4.

The domain of \( f \) is described with set-builder notation as \( f(D) = \{x \in \mathbb{R}, x \neq 4\} \).

The domain of \( f \) is described with interval notation as \( (\infty, 4) \cup (4, \infty) \).

Figure 1 summarizes this work.

b) Determine the zeros of \( f \)

The zeros of any function \( f \) are found by solving the equation resulting from \( f(x) = 0 \).

In this case we must solve the equation

Equation 1: \( \frac{2x - 3}{x - 4} = 0 \).

Multiply both sides of the equation by the denominator \( x - 4 \) to obtain

Equation 2: \( 2x - 3 = 0 \).

Observe that Equation 2 need not be equivalent to Equation 1. However, the solution set for Equation 2 contains the solution set for Equation 1. Therefore all solutions of Equation 2 are potential solutions of Equation 1, but it is necessary to test solutions of Equation 2 to determine if they are solutions of Equation 1. In the case of a rational equation, it is only necessary to insure that a potential solution not cause a 0 in a denominator.

The solution set for Equation 2 is \( \left\{ \frac{3}{2} \right\} \) and clearly \( \frac{3}{2} \) does not cause a 0 in the denominator of \( f \).

The zero of \( f \) is \( \frac{3}{2} \).

Figure 2
c) **Determine the vertical asymptotes of f**

Vertical asymptotes occur at zeros of the denominator which are not zeros of the numerator. The zero of the denominator is 4, which is not equal to $\frac{3}{2}$, the zero of the numerator. Therefore the vertical line $x = 4$ is a vertical asymptote of the graph of the function $f$.

d) **Determine and shade excluded regions**

Begin by sketching the vertical asymptotes on a rectangular coordinate system. Then sketch a vertical line through each zero of the function. The lines through the zeros of $f$ are construction lines and are not part of the graph of the function $f$. These vertical lines should be removed from the final graph of the function $f$. However, the vertical asymptotes are frequently included with the final graph and are frequently considered to be part of the graph.

The vertical lines through the zeros of $f$ combined with the vertical asymptotes of $f$ divide the coordinate system into a number of strips. In each individual strip the graph of $f$ will either be entirely above the x-axis or entirely below the x-axis. Test a single convenient domain element $k$ in each strip to determine whether $f(k) > 0$ or $f(k) < 0$.

- If $f(k) > 0$, the graph of $f$ will be above the x-axis in that strip and the portion of that strip which is below the x-axis is called an “excluded region” because the graph is excluded from that region. Shade all excluded regions.
- If $f(k) < 0$, the graph of $f$ will be below the x-axis in that strip and the portion of that strip which is above the x-axis is called an “excluded region” because the graph is excluded from that region. Shade all excluded regions.

In this example:

**Test 0:** $f(0) = \frac{2(0) - 3}{0 - 4} = \frac{-3}{-4} = \frac{3}{4} > 0$.

$f(0) > 0$ is the important observation.

Therefore the half-strip to the left of $\frac{3}{2}$ and below the x-axis is excluded.

**Test 3:** $f(3) = \frac{2(3) - 3}{3 - 4} = \frac{3}{-1} = -3 > 0$

$f(3) < 0$ is the important observation.

Therefore the half-strip between $\frac{3}{2}$ and 4 and above the x-axis is excluded.

**Test 5:** $f(5) = \frac{2(5) - 3}{5 - 4} = \frac{7}{1} = 7 > 0$

$f(5) > 0$ is the important observation.
e) Determine the horizontal asymptotes of \( f \)

The ratio of the degrees of the numerator and denominator of a rational function determine whether the function does or does not have a horizontal asymptote.

1. If the degree of the numerator is greater than the degree of the denominator, the function does not have a horizontal asymptote.
2. If the degree of the numerator is equal to the degree of the denominator, the function has a horizontal asymptote \( \frac{a_m}{b_n} \) where \( a_m \) is the leading coefficient of the numerator and \( b_n \) is the leading coefficient of the denominator.
3. If the degree of the numerator is less than the degree of the denominator, the function has the x-axis as its horizontal asymptote.

The numerator and denominator of the function \( f \) in this example have the same degree, so the horizontal asymptote is the horizontal line \( y = \frac{2}{1} = 2 \).

The horizontal asymptote is \( y = 2 \).

f) Determine where (if at all) the graph of \( f \) intersects its horizontal asymptote

The graph of \( f \) intersects its horizontal asymptote if and only if \( f(x) = 2 \) for some \( x \). Thus we find the solutions of the equation \( \frac{2x - 3}{x - 4} = 2 \). Multiply both sides of the equation by \( x - 4 \) to obtain \( 2x - 3 = 2x - 8 \). Add \(-2x\) to both sides of the equation to obtain \(-3 = -8\); a contradiction. Therefore the equation \( 2x - 3 = 2x - 8 \) has no solution and since the solution set for the equation \( \frac{2x - 3}{x - 4} = 2 \) is contained in the solution set for \( 2x - 3 = 2x - 8 \) the original equation \( \frac{2x - 3}{x - 4} = 2 \) has no solution and the graph of \( f \) does not intersect its horizontal asymptote.

The graph of \( f \) does not intersect its horizontal asymptote.

g) Sketch the graph of \( f \)

The graph of \( f \) as generated by a computer graphing utility is shown in Figure 6. The reader should verify that this graph matches the preceding analysis.
h) Determine where the graph of $f$ has horizontal tangents.

i) Determine where the graph of $f$ is increasing, decreasing.

j) Determine where the graph of $f$ has relative maxima and/or relative minima.

k) Determine where the graph of $f$ is concave down, up.

l) Modify the graph of $f$ as indicated by the calculus investigations.