Analysis of a Rational Function

Question:

Analyze and graph the rational function whose rule is

$$\textbf{Equation Q:}\quad f(x) = \frac{x^3 - x^2 - 5x - 3}{x^2 + 5x + 6} = \frac{N(x)}{D(x)}.$$

Analysis:

a) Determine the domain of f

General Principle: When the domain of a function is not explicitly stated, convention dictates that the domain is the largest subset of the real numbers for which the rule make sense (is defined).

In the case of a rational function the rule is defined for all real numbers except for zeros of the denominator D.

In this case the rule for f is defined for all real numbers except when

$D(x) = x^2 + 5x + 6 = (x + 3)(x + 2) = 0$.

The Zero Factor Property implies that solutions of this equation are -3 and -2 .

The domain of f is all real numbers except -3 and -2.

The domain of f is described with set-builder notation as

$$D_f = \{x \mid x \in \boldsymbol{R}, x \neq -3, \text{ or } x \neq -2\}.$$

The domain of f is described with interval notation as:

$$D_f = (-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$$

Figure 1 summarizes the conclusion about the domain of f.

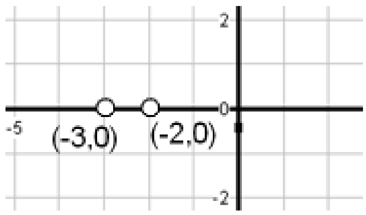

Figure 1

General Principle: Unless stated otherwise the domain of a rational function is all real numbers except the zeros of the denominator.

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b) Determine the zeros of f

General Principle: The zeros of any function h are found by solving the equation resulting from h(x) = 0.

In this case we must solve

$$\textbf{Equation 1:}\quad \frac{x^3 - x^2 - 5x - 3}{x^2 + 5x + 6} = 0$$

Multiply both sides of Equation 1 by the denominator $x^2 + 5x + 6$ to obtain

Equation 2: $x^3 - x^2 - 5x - 3 = 0$.

Observe that Equation 2 need not be equivalent to Equation 1. However, the solution set for Equation 2 contains the solution set for Equation 1.

Therefore all solutions of Equation 2 are potential solutions of Equation 1, but it is necessary to test solutions of Equation 2 to determine if they are solutions of Equation 1.

General Principle: In the case of a rational equation, it is only necessary to insure that a potential solution not cause a 0 in a denominator.

Because we are now interested in solving a polynomial equation it is helpful (and wise) to recall the following four equivalent statements.

General Principle: If h is a polynomial function whose rule is given by

$$h(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

then the following statements are equivalent.

1. **k is a real zero of the function h.**
2. **k is a solution of the polynomial equation $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0$.**
3. **x - k is a factor of the polynomial $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$.**
4. **(k, 0) is an x-intercept of the graph of the function h.**

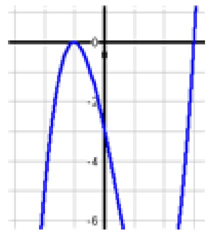
Equation 2 is a polynomial equation as listed in Item 2, so finding real solutions of Equation 2 is equivalent to finding real zeros of the corresponding polynomial function as specified in Item 1. Hence we may use the Rational Zeros Theorem.

Rational Zeros Theorem: If $\frac{p}{q}$ is a rational zero of a polynomial function h with integer coefficients, then p is a divisor of the constant term and q is a divisor of the leading coefficient.

In this case $p \in \{\pm 1, \pm 3\}$ and $q \in \{\pm 1\}$. Therefore the only possible rational zeros of N are $p \in \{\pm 1, \pm 3\}$.

A graphing utility produces the graph of N as shown at the right. From that graph of N it is clear that neither 1 nor –3 are reasonable possibilities.

Furthermore it appears that –1 is a zero of multiplicity 2 while 3 is a zero of multiplicity 1.



Graph of N

Polynomial division as shown at the right verifies that x + 1 is a factor of $x^3 - x^2 - 5x - 3$ and that the quotient is $x^2 - 2x - 3$.

Therefore we can write

$N(x) = x^3 - x^2 - 5x - 3 = (x + 1)(x^2 - 2x - 3) = (x + 1)^2 (x - 3)$.

The Zero Factor Property assures us that the zeros of N are indeed –1 (with multiplicity 2) and 3 (with multiplicity 1).

Since these zeros of N are not zeros of D, we conclude that they are zeros of f.

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ x + 1 & x^3 - x^2 - 5x - 3 \\ & \underline{x^3 + x^2} \\ & \quad -2x^2 - 5x - 3 \\ & \quad \underline{-2x^2 - 2x} \\ & \qquad -3x - 3 \\ & \qquad \underline{-3x - 3} \end{array}$$

General Principle: The zeros of a rational function are the zeros of the numerator which are not zeros of the denominator.

The zeros of f are –1 and 3.

The zeros of f are real and therefore correspond to x-intercepts (–1, 0) and (3, 0)

Because of their multiplicities, the graph of f will intersect but not cross the x-axis at (–1, 0) and it will cross the x-axis at (3, 0).

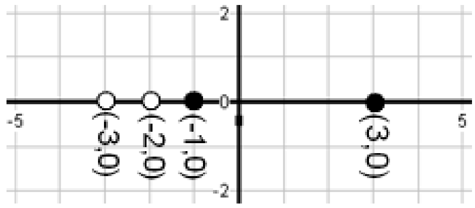


Figure 2

c) Determine the vertical asymptotes of f

General Principle: The vertical asymptotes for a rational function occur at the real zeros of the denominator which are not zeros of the numerator.

The graph of f has vertical asymptotes x = –3 and x = –2. Vertical asymptotes are shown in Figure 3

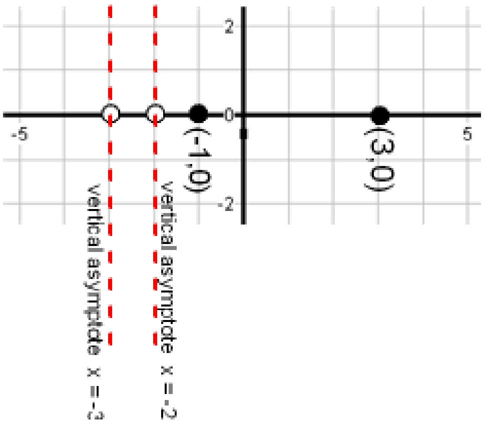


Figure 3
Shows domain, zeros, and vertical asymptotes

d) Determine and shade excluded regions

General Principle: The vertical asymptotes and vertical lines through the x-intercepts divide the plane into vertical strips. In each individual strip, the graph of the rational function will be entirely above the x-axis or entirely below the x-axis.

General Principle: A point (x, h(x)) on the graph of h is above the x-axis if and only if h(x) > 0.

General Principle: A point (x, h(x)) on the graph of h is below the x-axis if and only if h(x) < 0.

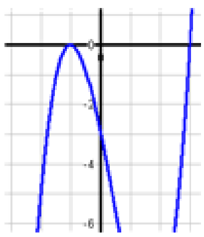
In many instances the graph produced by a graphing utility combined with the the three previous general principles will determine which half-strips are excluded regions. However, for the case of this rational function it is difficult to determine with a graphing utility. We therefore illustrate an alternate "array" method. Build an array with rows and columns as indicated in Array 1.

Observe that the rows correspond to the numerator N, denominator D, and the rational function f. Observe that the columns correspond to the strips of interest as determined by the zeros of the numerator and denominator.

	(-∞,-3)	(-3, -2)	(-2, -1)	(-1, 3)	(3, ∞)
N(x)					
D(x)					
f(x)					

Array 1

A cell will contain a + symbol if the function heading the row is positive in the interval specified at the top of the column. A cell will contain a – symbol if the function heading the row is negative in the interval specified at the top of the column. The graph of N and the graph of D (both produced with a graphing utility are used to easily determine where N(x) > 0, N(x) < 0, D(x) > 0, and D(x) < 0. This produces two rows of the array as shown in Array 2.

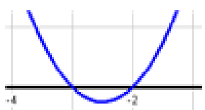


Graph of N

:

	(-∞,-3)	(-3, -2)	(-2, -1)	(-1, 3)	(3, ∞)
N(x)	–	–	–	–	+
D(x)	+	–	+	+	+
f(x)					

Array 2



Graph of D

Finally, consideration of the sign of the quotient of N(x) and D(x) produces the sign of f(x) in each of the intervals (strips) as detailed in Array 3.

	(-∞,-3)	(-3, -2)	(-2, -1)	(-1, 3)	(3, ∞)
N(x)	−	−	−	−	+
D(x)	+	−	+	+	+
f(x)	−	+	−	−	+

Array 3

We now can observe from Array 3 that:

$f(x) > 0$ if $x \in (-3, -2) \cup (3, \infty)$

$f(x) < 0$ if $x \in (-\infty, -3) \cup (-2, -1) \cup (-1, 3)$

Previous work in Parts C and D show

$f(x) = 0$ if $x \in \{-1, 3\}$ and

f(x) does not exist (f is undefined) if $x \in \{-3, -2\}$

General Principle: For any function h and any real number x there are exactly two possibilities:

a) h(x) exists --- h is defined

b) h(x) does not exist --- h is not defined

For those real numbers for which h(x) exists The Law of Trichotomy implies there are exactly three possibilities:

1) $h(x) > 0$

2) $h(x) = 0$

3) $h(x) < 0$.

Results of the analysis to this point are displayed in Figure 4. Regions of the plane where the graph of f cannot exist (called excluded regions) are shaded.

The y-intercept $(0, f(0)) = \left((0, -\frac{1}{2}\right)$ is also shown.

General Principle: The y-intercept of the graph of a function h is (0, h(0)).

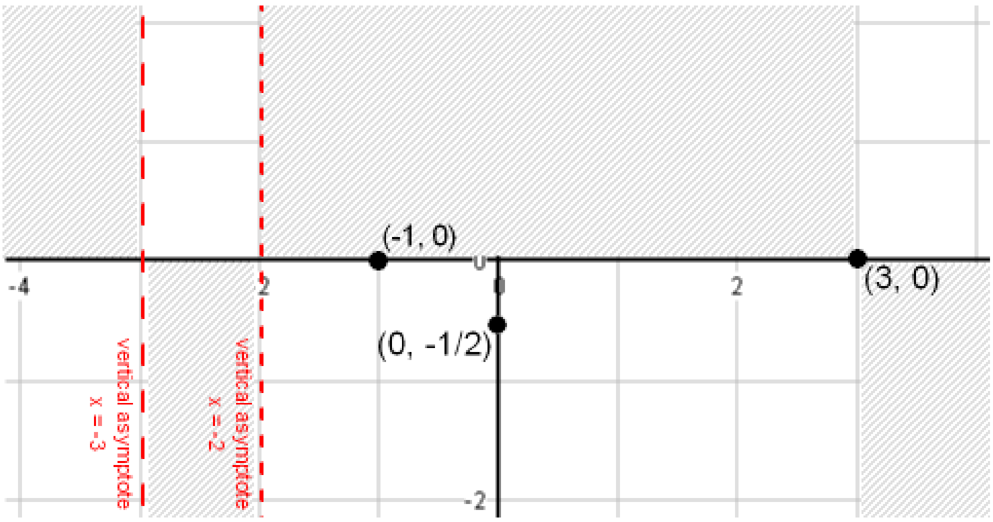


Figure 4

e) Determine the horizontal asymptotes of f

General Principle: Suppose f is a rational function whose rule is

$$h(x) = \frac{a_m x^m + a_{m-1}x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0}$$

1. If m < n, the line y = 0 (the x-axis) is a horizontal asymptote.

2. If m = n, the line $y = \frac{a_m}{b_n}$ is a horizontal asymptote.

3. if m > n, there is no horizontal asymptote.

The rational function f (Equation Q) in this example has a numerator with degree 3 and a denominator with degree 2.

Therefore there is no horizontal asymptote.

f) Determine slant or curvilinear asymptotes of f

General Principle: For any rational function $h(x) = \frac{N(x)}{D(x)}$, the Division Algorithm for Polynomials implies there are unique polynomials Q(x) and R(x) such that

N(x) = Q(x)D(x) + R(x) and 0 ≤ degree of R(x) < degree of D(x).

The quotient Q(x) is an asymptote for the rational function h.

To understand why Q(x) is an asymptote for f(x) note that N(x) = Q(x)D(x) + R(x) can be rewritten as $\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$. Because the degree of R(x) is less than the degree of D(x), it follows that the horizontal asymptote of the function $\frac{R(x)}{D(x)}$ is y = 0. Thus $\frac{R(x)}{D(x)} \to 0$ as $x \to \pm\infty$ which implies $f(x) \to Q(x)$ as $x \to \pm\infty$.

Consequently Q(x) is an asymptote for the function f.

The indicated division in the rule for f (in Equation Q) for this example is shown at the right.

$$\begin{array}{r} x - 6 \\ x^2 + 5x + 6 \,\overline{\big)\, x^3 - x^2 - 5x - 3} \\ \underline{x^3 + 5x^2 + 6x} \\ -6x^2 - 11x - 3 \\ \underline{-6x^2 - 30x - 36} \\ 19x + 33 \end{array}$$

The line y = x – 6 is an asymptote for the rational function f.

g) Determine where (if at all) the graph of f intersects its asymptote

General Principle: To find the intersection of graphs of two functions h and g it is necessary and sufficient to solve the equation resulting from h(x) = g(x).

In our case we must solve the equation $\frac{x^3 - x^2 - 5x - 3}{x^2 + 5x + 6} = f(x) = g(x) = x - 6$.

Multiply both sides by the denominator, collect terms and solve the resulting linear equation to obtain x = $-\frac{33}{19}$. Then compute the second coordinate of the point of intersection by substituting x = $-\frac{33}{19}$ into y = x - 6 to obtain $\frac{-147}{19}$. For convenience when plotting the point, the coordinates may be converted to mixed numbers or decimals as $\left(-1\frac{14}{19},-7\frac{14}{19}\right)$ or (-1.7, -7.8)

The graph of f intersects its asymptote y = x – 6 at the point $\left(-\frac{33}{19},-\frac{147}{19}\right)$

h) Sketch the graph of f

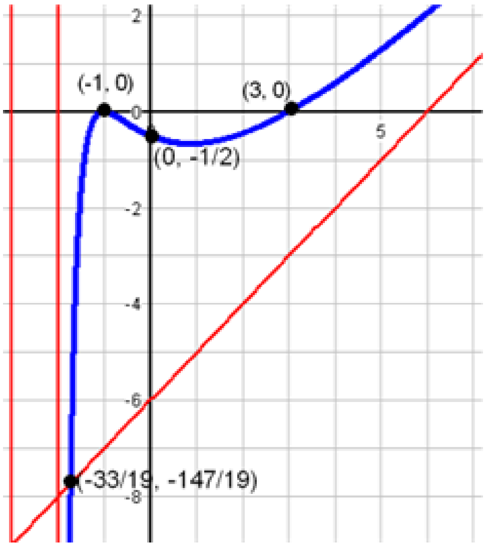


Figure 6
Near the origin

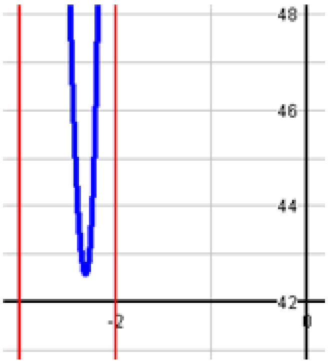


Figure 7
Between the vertical asymptotes
Note the position on the y-axis

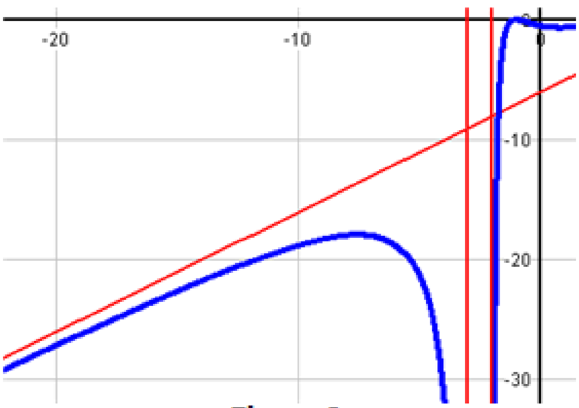


Figure 8
The portion to left of the vertical asymptotes

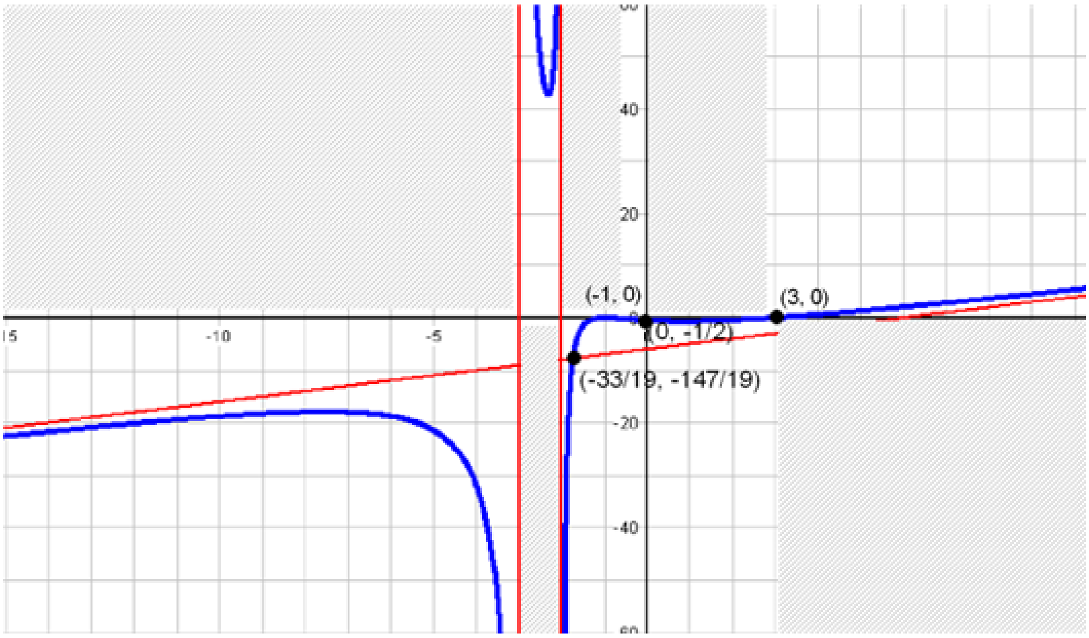


Figure 5
Scale on axis are not the same
To show all three segments of the graph

In Figure 9 the asymptotic behavior along the slant asymptote is quite evident.

However behavior near the origin and at the x-intercepts is obscured when the graph is viewed at this scale.

Figure 9
Scales are the same on the two axis